Stability of the space of Automorphic Forms under Local Deformations of the Lattice

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Mathematical Subject Classification

11F12 (Primary) , 30F10 (Secondary) .

Keywords

Automorphic and cusp forms on the upper half plane, local deformation, ringed spaces, compact Riemann surfaces and holomorphic line bundles, Teichmüller space.

Abstract

First we explain the concept of local deformation over a 'parameter' algebra $\mathcal{P}$, in particular the notion of a $\mathcal{P}$-lattice in a Lie group. Purpose of this article is to define the spaces $M_k(\Upsilon)$ and $S_k(\Upsilon)$ of automorphic resp. cusp forms on the upper half plane H for a $\mathcal{P}$- (!) lattice Υ of $SL(2,\mathbb{R})$ and to investigate their structure. It turns out that in almost all cases the spaces $M_k(\Upsilon)$ and $S_k(\Upsilon)$ are free modules over the complexified $\mathcal{P}$ of rank equal to the dimension of the spaces of automorphic resp. cusp forms for the body $\Gamma := \Upsilon^\#$, which is the associated ordinary lattice in $SL(2,\mathbb{R})$. In other words almost every automorphic resp. cusp form admits an 'adaption' to local deformations of the lattice. This is shown by giving the quotient $\Upsilon\backslash H$ together with the cusps of $\Gamma\backslash H$ the structure of a $\mathcal{P}$- Riemann surface and writing the spaces of automorphic resp. cusp forms as global sections of holomorphic $\mathcal{P}$- (!) line bundles on $\Upsilon\backslash H \cup \{$ cusps of $\Gamma\backslash H\}$.

Introduction

First of all let us discuss the concept of local deformation. A rough explanation is the following: Vary the 'data' describing a classical object, for example the glueing data of local charts defining a smooth manifold, let them depend on 'parameters' generating a local commutative algebra $\mathcal{P}$. In practice it is not necessary to specify the parameters since all information is already encoded in this algebra. Here we consider the case of an finite dimensional algebra $\mathcal{P}$ whose unique maximal ideal $\mathcal{I}$ is nilpotent, and $\mathcal{P}/\mathcal{I} \simeq \mathbb{R}$. This case obviously lies in-between the extremal poles of infinitesimal deformation, which means $\mathcal{I}^2 = 0$, and arbitrary local deformation. So it is not surprising that many proofs in this context use the techniques coming from infinitesimal deformation in combination with induction over the power annihilating $\mathcal{I}$. An object $\mathcal{O}$ whose data depend on the 'parameters' generating $\mathcal{P}$ will be called a $\mathcal{P}$-object. It is necessarily a local deformation of a classical object $\mathcal{O}^\#$, called its body. We will see that in general this gives whole body functors from the categories of $\mathcal{P}$-objects to their classical counterparts induced by the canonical projection $\mathcal{P} \to \mathcal{P}/\mathcal{I} \simeq \mathbb{R}$, heuristically the 'set all the parameters to 0 ' functors.

Natural questions in the framework of local deformation are the following: Do we really get more objects when we allow $\mathcal{P}$-ones, in other words do certain objects allow non-trivial local deformation or are they completely rigid? Can we adapt 'functions' on a classical object O to local deformations of O ? How can one classify all $\mathcal{P}$-objects with given body?

In this article we study $\mathcal{P}$-lattices in $SL(2,\mathbb{R})$ acting on the upper half plane $H \subset \mathbb{C}$, in other words local deformation of the natural embedding of a given lattice $\Gamma \hookrightarrow SL(2,\mathbb{R})$ as group homomorphism and we want to investigate the spaces of automorphic and cusp forms for these $\mathcal{P}$-lattices. The theory of automorphic forms for classical lattices is already well-established. Let $\Gamma \sqsubset SL(2,\mathbb{R})$ be a lattice. Then we have an asymptotic formula

$$\dim M_k(\Gamma), \dim S_k(\Gamma) \sim \frac{k}{4\pi} \mathrm{vol}\ (\Gamma\backslash H)$$

for the dimension of the spaces $M_k(\Gamma)$ and $S_k(\Gamma)$ of automorphic resp. cusp forms for Γ of high weight k , and this is one of the most beautiful applications of the theory of holomorphic line bundles on compact RIEMANN surfaces. Now in the case of a $\mathcal{P}$-lattice Υ in $SL(2,\mathbb{R})$ with body Γ it would be nice if every form $f \in M_k(\Gamma)$ would allow an adaption $\widetilde{f} \in M_k(\Upsilon)$ to Υ

having f as body, because this is equivalent to the stability of $M_k(\Gamma)$ under local deformations of Γ . We will show that this is precisely equivalent to $M_k(\Upsilon) \simeq M_k(\Gamma) \otimes \mathcal{P}^{\mathbb{C}}$ as $\mathcal{P}^{\mathbb{C}}$-modules and, as the main result of this paper, is indeed true except in the case where $k = 1$ and $\Gamma \backslash H \cup \{$ cusps of $\Gamma \backslash H\}$ is of genus $g \geq 2$, see theorems 5.3 and 5.7 for even resp. odd $k \in \mathbb{N}$.

Imitating the classical theory, the article is organized as follows: The general concept of $\mathcal{P}$-manifolds and $\mathcal{P}$- vector bundles is introduced in section 1 , and section 2 treats the basic definitions of $\mathcal{P}$-lattices and associated automorphic and cusp forms in the case of $SL(2, \mathbb{R})$ acting on H . In section 3 after fixing a $\mathcal{P}$-lattice Υ of $SL(2, \mathbb{R})$ with body $\Gamma := \Upsilon^\#$ we construct a $\mathcal{P}$- Riemann surface $\mathcal{X}$ which is in some sense a compactification of $\Upsilon \backslash H$ in analogy to $X := \Gamma \backslash H \cup \{$ cusps of $\Gamma \backslash H\}$, which will be the body of $\mathcal{X}$. In section 4 we do some elementary Teichmüller theory, more precisely we prove that any $\mathcal{P}$- Riemann surface $\mathcal{X}$ with compact body $X := \mathcal{X}^\#$ is represented by a $\mathcal{P}$-point of the Teichmüller space whose body represents X , see theorem 4.3 . This result is of course of general interest since it gives a complete solution for the classification problem of $\mathcal{P}$- Riemann surfaces with given compact body and is therefore given in greatest possible generality. In section 5 we use all our knowledge obtained so far to prove the main theorems of this article, theorems 5.3 and 5.7 . Finally section 6 deals with the special case $\Upsilon^\# = SL(2, \mathbb{Z})$.

Acknowledgement: I have to thank M. Schlichenmaier for many helpful comments during the writing process and the Fonds National de la Recherche Luxembourg for funding my research stay at Luxembourg university.

1 $\mathcal{P}$-manifolds

For the whole article let $\mathcal{P}$ be a finite dimensional real unital commutative algebra with a unital algebra projection $^\# : \mathcal{P} \to \mathbb{R}$ and the unique maximal ideal $\mathcal{I} := \mathrm{Ker}\ ^\# \lhd \mathcal{P}$ such that $\mathcal{I}^N = 0$ for some $N \in \mathbb{N}$. $^\#$ is called the body map of $\mathcal{P}$.

Examples 1.1

(i) The even part $\mathcal{P} := \bigwedge \left(\mathbb{R}^{N-1}\right)_0$ of an exterior algebra,

(ii) the polynomial algebra $\mathcal{P} := \mathbb{R}[X] \big/ \big(X^N = 0\big)$ with cut off.

Defining the category of $\mathcal{P}$-manifolds will be in terms of ringed spaces, the real and complex case treated simultaneously. Therefore let $\mathcal{P}^{\mathbb{C}}$ and $\mathcal{I}^{\mathbb{C}}$ denote the complexifications of $\mathcal{P}$ resp. $\mathcal{I}$.

Definition 1.2 ($\mathcal{P}$-manifolds and $\mathcal{P}$-points)

(i) Let M be a real smooth (complex) manifold of dimension n , and $\mathcal{S}$ be a sheaf of unital commutative $\mathcal{P}$- ($\mathcal{P}^{\mathbb{C}}$-) algebras over M . Then the ringed space $\mathcal{M} := (M, \mathcal{S})$ is called a real (complex) $\mathcal{P}$-manifold of dimension n if and only if locally $\mathcal{S} \simeq \mathcal{C}_M^\infty \otimes \mathcal{P}$ ($\mathcal{S} \simeq \mathcal{O}_M \otimes \mathcal{P}^{\mathbb{C}}$) . $M := \mathcal{M}^\#$ is called the body of $\mathcal{M}$. If M is a complex manifold of dimension $n = 1$ then $\mathcal{M}$ is called a $\mathcal{P}$- Riemann surface.

(ii) Let $\mathcal{M} = (M, \mathcal{S})$ and $\mathcal{N} = (N, \mathcal{T})$ be two real (complex) $\mathcal{P}$-manifolds. A $\mathcal{P}$-morphism between $\mathcal{M}$ and $\mathcal{N}$ is a morphism from $\mathcal{M}$ two $\mathcal{N}$ as ringed spaces, more precisely a collection $\Phi := \left(\varphi, (\phi_V)_{V \subset N \text{ open}}\right)$ where $\varphi : M \to N$ is a smooth (holomorphic) map, and all $\phi_V : \mathcal{T}(V) \to \mathcal{S}\left(\varphi^{-1}(V)\right)$ are unital $\mathcal{P}$- ($\mathcal{P}^{\mathbb{C}}$-) algebra homomorphisms such that for all $W \subset V \subset N$ open

$$
\begin{array}{ccc}
\mathcal{T}(V) & \xrightarrow{\phi_V} & \mathcal{S}\left(\varphi^{-1}(V)\right) \\
|_W \downarrow & \circlearrowleft & \downarrow |_{\varphi^{-1}(W)} \\
\mathcal{T}(W) & \xrightarrow[\phi_W]{} & \mathcal{S}\left(\varphi^{-1}(W)\right)
\end{array} .
$$

We write $\Phi : \mathcal{M} \to_{\mathcal{P}} \mathcal{N}$ and $f(\Phi) := \phi_V(f) \in \mathcal{S}\left(\varphi^{-1}(V)\right)$ for all $V \subset N$ open and $f \in \mathcal{T}(V)$. $\Phi^\# := \varphi$ is called the body of Φ .

(iii) Let $a = \left(a^\#, (a_V)_{V \subset N \text{ open}}\right) : \mathcal{M} \to_{\mathcal{P}} \mathcal{N}$ be a $\mathcal{P}$-morphism from the real (complex) $\mathcal{P}$-manifold $\mathcal{M} := (\{0\}, \mathcal{P})$ ($\mathcal{M} := \left(\{0\}, \mathcal{P}^{\mathbb{C}}\right)$) to the real (complex) $\mathcal{P}$-manifold $\mathcal{N}$. Then a is called a $\mathcal{P}$-point of $\mathcal{N}$. Its body $a^\# : \{0\} \hookrightarrow \mathcal{N}^\#$ will always be identified with the usual point $a^\#(0) \in \mathcal{N}^\#$. We write $a \in_{\mathcal{P}} \mathcal{N}$ and $f(a) := a_V(f)$ for all $V \subset N$ open with $a^\# \in V$ and $f \in \mathcal{T}(V)$. If $\mathcal{O}$ is another $\mathcal{P}$-manifold and $\Phi : \mathcal{N} \to_{\mathcal{P}} \mathcal{O}$ a $\mathcal{P}$-morphism then we write $\Phi(a) := \Phi \circ a \in_{\mathcal{P}} \mathcal{O}$.

Let us collect some basic properties of $\mathcal{P}$-manifolds, they will be implicitly used later:

(i) Let $\mathcal{M} = (M, \mathcal{S})$ be a real (complex) $\mathcal{P}$-manifold. Then since $\mathcal{C}_M^\infty$ ($\mathcal{O}_M$) has no other unital sheaf automorphisms than id , $^\# : \mathcal{P} \to \mathbb{R}$ induces a body map $^\# : \mathcal{S} \to \mathcal{C}_M^\infty$ ($^\# : \mathcal{S} \to \mathcal{O}_M$) , which is a projection of sheaves of real (complex) unital algebras. The kernel of $^\#$ is precisely $\mathcal{I}\mathcal{S}$ ($\mathcal{I}^{\mathbb{C}}\mathcal{S}$), and we have a canonical sheaf isomorphism $\mathcal{C}_M^\infty \otimes \mathcal{I}^n \simeq \mathcal{I}^n\mathcal{S}$ ($\mathcal{O}_M \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^n \simeq \left(\mathcal{I}^{\mathbb{C}}\right)^n \mathcal{S}$) whenever $n \in \mathbb{N}$ such that $\mathcal{I}^{n+1} = 0$.

Now let $\mathcal{N} := (N, \mathcal{T})$ be another real (complex) $\mathcal{P}$-manifold and

$\Phi := \left(\varphi, (\phi_V)_{V \subset N \text{ open}}\right)$ a $\mathcal{P}$ morphism from $\mathcal{M}$ to $\mathcal{N}$. Then automatically

$$\begin{array}{ccc} \mathcal{T}(V) & \xrightarrow{\phi_V} & \mathcal{S}\left(\varphi^{-1}(V)\right) \\ \# \downarrow & \circlearrowleft & \downarrow \# \\ \mathcal{C}^\infty(V) & & \mathcal{C}^\infty\left(\varphi^{-1}(V)\right) \\ (\quad \mathcal{O}(V) \quad) & \xrightarrow[f \mapsto f \circ \varphi|_{\varphi^{-1}(V)}]{} & (\quad \mathcal{O}\left(\varphi^{-1}(V)\right) \quad) \end{array}$$

for all $V \subset N$ open, and for all $f \in \mathcal{T}(V)$ we call
$\phi_V(f) = f(\Phi) \in \mathcal{S}\left(\varphi^{-1}(V)\right)$ the pullback of f under Φ .

(ii) Every usual real smooth (complex) manifold M can be regarded as a real (complex) $\mathcal{P}$-manifold identifying M with the ringed space $(M, \mathcal{C}_M^\infty \otimes \mathcal{P})$ ($\left(M, \mathcal{O}_M \otimes \mathcal{P}^\mathbb{C}\right)$) , and every usual smooth (holomorphic) map between real smooth (complex) manifolds can be regarded as a $\mathcal{P}$-morphism between them.

(iii) Let $\Phi = \left(\varphi, (\phi_V)_{V \subset N \text{ open}}\right) : \mathcal{M} \to_\mathcal{P} \mathcal{N}$ be a $\mathcal{P}$-morphism between the real (complex) $\mathcal{P}$-manifolds $\mathcal{M} = (M, \mathcal{S})$ and $\mathcal{N} = (N, \mathcal{T})$. Then it is an isomorphism iff φ is bijective, and in this case Φ is called a $\mathcal{P}$-isomorphism.

If φ is an immersion then for all $a \in M$ there exists an open neighbourhood V of $\varphi(a)$ in N such that $\phi_V : \mathcal{T}(V) \to \mathcal{S}\left(\varphi^{-1}(V)\right)$ is surjective. Φ is called a $\mathcal{P}$-embedding iff φ is an injective immersion and so a smooth (holomorphic) embedding of real smooth (complex) manifolds, and in this case we write $\Phi : \mathcal{M} \hookrightarrow_\mathcal{P} \mathcal{N}$.

Furthermore if φ is surjective and for all $a \in M$ there exists an open neighbourhood V of $\varphi(a)$ in N such that $\phi_V : \mathcal{T}(V) \to \mathcal{S}\left(\varphi^{-1}(V)\right)$ is injective then Φ is called a $\mathcal{P}$-projection.

(iv) Let $\Phi = \left(\varphi, (\phi_V)_{V \subset N \text{ open}}\right) : \mathcal{M} \hookrightarrow_\mathcal{P} \mathcal{N}$ be a $\mathcal{P}$-embedding from the real (complex) $\mathcal{P}$-manifold $\mathcal{M} = (M, \mathcal{S})$ into the real (complex) $\mathcal{P}$-manifold $\mathcal{N} = (N, \mathcal{T})$. Since $\varphi : M \hookrightarrow N$ is an embedding we can regard M as a real smooth (complex) submanifold of N and φ as the canonical inclusion. So Φ induces a sheaf projection $\pi : \mathcal{T}|_M \to \mathcal{S}$ such that for all $V \subset N$ open

$$\begin{array}{ccccc} \mathcal{T}(V) & & \xrightarrow{\phi_V} & & \mathcal{S}(V\cap M) \\ & |_{V\cap M} \searrow & \circlearrowleft & \nearrow_{\pi_{V\cap M}} & \\ & & \mathcal{T}|_N(V\cap M) & & \end{array} .$$

In particular there is a 1-1 correspondence between $\mathcal{P}$-points $a = \left(a^\#, (a_V)_{V\subset N \text{ open}}\right)$ of $\mathcal{N}$ and pairs $\left(a^\#, a_{a^\#}\right)$ where $a^\# \in \mathcal{N}^\#$ and $a_{a^\#} : \mathcal{T}_{a^\#} \to \mathcal{P}$ ($a_{a^\#} : \mathcal{T}_{a^\#} \to \mathcal{P}^{\mathbb{C}}$) is an epimorphism of $\mathcal{P}$- ($\mathcal{P}^{\mathbb{C}}$-) algebras, where $\mathcal{T}_{a^\#}$ denotes the stalk of $\mathcal{T}$ at $a^\# \in N$, such that for all $V \subset N$ open if $a^\# \in V$ then $a_V = a_{a^\#} \circ |_{a^\#}$, where $|_{a^\#} : \mathcal{T}(V) \to \mathcal{T}_{a^\#}$ denotes the canonical projection, and otherwise $a_V \equiv 0$.

(v) The local models of real (complex) $\mathcal{P}$-manifolds are the usual open sets $U \subset \mathbb{R}^m$ ($U \subset \mathbb{C}^m$) regarded as real (complex) $\mathcal{P}$-manifolds together with $\mathcal{P}$-morphisms between them. Let $\mathcal{M} = (M, \mathcal{S})$ be an m-dimensional real (complex) $\mathcal{P}$-manifold and $V \subset \mathbb{R}^n$ ($V \subset \mathbb{C}^n$) be open. Then one can show that there is a 1-1 correspondence between $\mathcal{P}$-morphisms from U to V and n-tuples $(f_1, \dots, f_n) \in \mathcal{S}(M)^{\oplus n}$ such that $\left(f_1^\#(u), \dots, f_n^\#(u)\right) \in V$ for all $u \in M$ given as follows:

To a $\mathcal{P}$-morphism $\Phi := \left(\varphi, (\phi_W)_{W\subset V \text{ open}}\right) : \mathcal{M} \to_{\mathcal{P}} V$ we associate the tuple $(\phi_V(x_1), \dots, \phi_V(x_n))$ ($(\phi_V(z_1), \dots, \phi_V(z_n))$) , where $x_1, \dots, x_n \in \mathcal{C}^\infty(V)$ ($z_1, \dots, z_n \in \mathcal{O}(V)$) denote the coordinate functions on V .

Conversely to a tuple $(f_1, \dots, f_n)$ we associate the $\mathcal{P}$-morphism $\Phi := \left(\varphi, (\phi_W)_{W\subset V \text{ open}}\right) : \mathcal{M} \to_{\mathcal{P}} V$, where $\varphi : M \to V$, $u \mapsto \left(f_1^\#(u), \dots, f_n^\#(u)\right)$ and

$$\begin{aligned} \phi_W : \mathcal{C}^\infty(W) \otimes \mathcal{P} &\to \mathcal{S}\left(\varphi^{-1}(W)\right) \\ \left(\quad \mathcal{O}(W) \otimes \mathcal{P}^{\mathbb{C}} \right. &\to \left. \mathcal{S}\left(\varphi^{-1}(W)\right) \quad \right), \\ h \mapsto \sum_{\mathbf{k}\in\mathbb{N}^n} \frac{1}{\mathbf{k}!} &\left(\left(\partial^{\mathbf{k}} h\right) \circ \left(\varphi|_{\varphi^{-1}(W)}\right)\right) \times \\ &\times \left(f_1 - f_1^\#\right)^{k_1} \cdots \left(f_n - f_n^\#\right)^{k_1}\Big|_{\varphi^{-1}(W)} \end{aligned}$$

for all $W \subset V$ open, which is nothing but the formal TAYLOR expansion of the expression $h(f_1, \dots, f_n)$.

In particular one can identify the $\mathcal{P}$-points $\mathbf{a}$ of V with the tuples $(a_1, \dots, a_n) \in \mathcal{P}^{\oplus n}$ ($(a_1, \dots, a_n) \in \left(\mathcal{P}^{\mathbb{C}}\right)^{\oplus n}$) such that $\left(a_1^\#, \dots, a_n^\#\right) \in V$ as follows:

To a $\mathcal{P}$-point $\mathbf{a} \in_{\mathcal{P}} V$ one assigns the tuple $(\mathbf{a}_V(x_1), \ldots, \mathbf{a}_V(x_n))$ (resp. $(\mathbf{a}_V(z_1), \ldots, \mathbf{a}_V(z_n))$),

and conversely to a tuple $(a_1, \ldots a_n)$ one assigns $\mathbf{a} = \left(\mathbf{a}^{\#}, \mathbf{a}_{\mathbf{a}^{\#}}\right)$, where $\mathbf{a}^{\#} = \left(a_1^{\#}, \ldots a_n^{\#}\right) \in V$ and

$$\begin{aligned} \mathbf{a}_{\mathbf{a}^{\#}} : \mathcal{C}_V^{\infty}|_{\mathbf{a}^{\#}} \otimes \mathcal{P} \to \mathcal{P} \qquad & \left(\quad \mathcal{O}_V|_{\mathbf{a}^{\#}} \otimes \mathcal{P}^{\mathbb{C}} \to \mathcal{P}^{\mathbb{C}} \quad \right), \\ h \mapsto \sum_{\mathbf{k} \in \mathbb{N}^n} \frac{1}{\mathbf{k}!} \partial^{\mathbf{k}} h \left(\mathbf{a}^{\#}\right) & \left(a_1 - a_1^{\#}\right)^{k_1} \cdots \left(a_n - a_n^{\#}\right)^{k_1} , \end{aligned}$$

which is again nothing but the formal TAYLOR expansion of the expression $h(a_1, \ldots, a_n)$.

Now let $\mathcal{M} = (M, \mathcal{S})$ and $\mathcal{N} = (N, \mathcal{T})$ be real (complex) $\mathcal{P}$-manifolds and $V \subset \mathbb{R}^k$ ($V \subset \mathbb{C}^k$) open. Let $\Phi : \mathcal{M} \to_{\mathcal{P}} \mathcal{N}$ and $\Psi : \mathcal{N} \to_{\mathcal{P}} V$ be $\mathcal{P}$-morphisms and Ψ be given by the tuple $(f_1, \ldots, f_k) \in \mathcal{T}(N)^{\oplus k}$. Then $\Psi \circ \Phi$ is given by the tuple $(f_1(\Phi), \ldots, f_k(\Phi)) \in \mathcal{S}(M)^{\oplus k}$. In particular if $a \in_{\mathcal{P}} \mathcal{N}$ then $\Psi(a) \in_{\mathcal{P}} V$ is given by the tuple $(f_1(a), \ldots, f_k(a)) \in \mathcal{P}^{\oplus k}$ ($(f_1(a), \ldots, f_k(a)) \in \left(\mathcal{P}^{\mathbb{C}}\right)^{\oplus k}$).

Of course given a real (complex) $\mathcal{P}$-manifold $\mathcal{M} = (M, \mathcal{S})$ of dimension n for each $a \in M$ there exists an open neighbourhood $U \subset M$ of a , $V \subset \mathbb{R}^n$ ($V \subset \mathbb{C}^n$) open and a $\mathcal{P}$-isomorphism from $(U, \mathcal{S}|_U)$ to $(V, \mathcal{C}_V^{\infty} \otimes \mathcal{P})$ ($(V, \mathcal{O}_V \otimes \mathcal{P})$) . Such a $\mathcal{P}$-isomorphism is called a local $\mathcal{P}$-chart of $\mathcal{M}$ at a . Two local $\mathcal{P}$-charts $V_i, V_j \subset \mathbb{R}^n$ ($V_i, V_j \subset \mathbb{C}^n$) 'glue' together via a $\mathcal{P}$-glueing data, given as a $\mathcal{P}$-isomorphism $\Phi_{ij} : V_{ij} \to_{\mathcal{P}} V_{ji}$ between the overlaps $V_{ij} \subset V_i$ and $V_{ji} \subset V_j$ open. The body then will be given by the same local charts with glueing data $\Phi_{ij}^{\#}$. Observe that in general one can not specify ordinary points of a real (complex) $\mathcal{P}$-manifold, only $\mathcal{P}$-points. However, given a $\mathcal{P}$-point $a \in_{\mathcal{P}} \mathcal{M}$ there always exists a local $\mathcal{P}$-chart of $\mathcal{M}$ at $a^{\#}$ mapping a to a usual point of $\mathbb{R}^n$ ($\mathbb{C}^n$).

Now let $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$ be open. Then U and V can be regarded as open subsets of $\mathbb{R}^{2m}$ resp. $\mathbb{R}^{2n}$, and using the 1-1 correspondence from above we see that every $\mathcal{P}$-morphism from U to V regarded as complex open sets is at the same time a $\mathcal{P}$-morphism from U to V as real open sets, and so we get a whole 'forget' functor from the category of complex $\mathcal{P}$-manifolds to the one of real $\mathcal{P}$-manifolds.

(vi) Let $\mathcal{M}$ be a real $\mathcal{P}$-manifold. Then there exists a $\mathcal{P}$-isomorphism $\Phi : \mathcal{M}^{\#} \to_{\mathcal{P}} \mathcal{M}$ such that $\Phi^{\#} = \mathrm{id}$. This can be shown by induction on N using $H^1\left(\mathcal{M}^{\#}, T\mathcal{M}^{\#}\right) = 0$, and it is nothing but the rigidity of smooth manifolds under local deformation.

(vii) Let $\mathcal{M} = (M, \mathcal{S})$ be a real (complex) $\mathcal{P}$-manifold of dimension m , $\Phi := \left(\varphi, (\Phi_V)_{V \subset M \text{ open}}\right) : \mathcal{M} \to_{\mathcal{P}} \mathbb{R}^n$ ($\mathcal{M} \to_{\mathcal{P}} \mathbb{C}^n$) be a $\mathcal{P}$-morphism such that $D\Phi^{\#}$ is surjective at every point of M and finally $\mathbf{a} \in_{\mathcal{P}} \mathbb{R}^n$ ($\mathbf{a} \in_{\mathcal{P}} \mathbb{C}^n$) . Then we can define the preimage $\Phi^{-1}(\mathbf{a})$ of $\mathbf{a}$ under Φ as a real (complex) $\mathcal{P}$-manifold $(N, \mathcal{T})$ of dimension $m - n$ as follows: Its body is $N := \varphi^{-1}\left(\mathbf{a}^{\#}\right)$, and the sheaf $\mathcal{T}$ is given by

$$\mathcal{T} := \mathcal{S}|_N \,/\, \mathfrak{m} \,,$$

where $\mathfrak{m} \lhd \mathcal{S}|_N$ is the ideal sheaf generated by all $\phi_V(f)$, where $V \subset \mathbb{R}^n$ ($V \subset \mathbb{C}^n$) open and $f \in \mathcal{C}^\infty(V)$ ($f \in \mathcal{O}(V)$) such that $f(\mathbf{a}) = 0$ if $\mathbf{a}^{\#} \in V$. If Φ is given by the tuple $(f_1, \dots, f_n) \in \mathcal{S}(M)^{\oplus n}$ and $\mathbf{a}$ by $(a_1, \dots, a_n) \in \mathcal{P}^{\oplus n}$ ($(a_1, \dots, a_n) \in \left(\mathcal{P}^{\mathbb{C}}\right)^{\oplus n}$) then $\mathfrak{m}$ is generated by $f_1 - a_1, \dots, f_n - a_n$. The $\mathcal{P}$-morphism $I := \left(i, (\pi_U)_{U \subset M \text{ open}}\right) : \mathcal{N} \hookrightarrow_{\mathcal{P}} \mathcal{M}$, where $i : N \hookrightarrow M$ is the canonical inclusion, and $\pi_U : \mathcal{S}(U) \to \mathcal{T}(U \cap N)$, $U \subset M$ open, denote the canonical projections, is a $\mathcal{P}$-embedding called the canonical inclusion of $\mathcal{N}$ into $\mathcal{M}$.

Let $\mathcal{O}$ be another real (complex) $\mathcal{P}$-manifold. Then there exists a 1-1 correspondence between the $\mathcal{P}$-morphisms $\Psi : \mathcal{O} \to_{\mathcal{P}} \mathcal{N}$ and the $\mathcal{P}$-morphisms $\Xi : \mathcal{O} \to_{\mathcal{P}} \mathcal{M}$ having

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\Xi} & \mathcal{M} \\ \mathrm{Pr} \downarrow & \circlearrowleft & \downarrow \Phi \\ \{0\} & \xrightarrow[\mathbf{a}]{} & \mathbb{R}^n \end{array} ,$$

where $\mathrm{Pr} : \mathcal{O} \to_{\mathcal{P}} \{0\}$ denotes the canonical $\mathcal{P}$-projection. It is given by the assignment $\Psi \mapsto I \circ \Psi$.

In particular we can identify the $\mathcal{P}$-points of $\mathcal{N}$ with the $\mathcal{P}$-points $b \in_{\mathcal{P}} \mathcal{M}$ of $\mathcal{M}$ having $\Phi(b) = \mathbf{a}$.

(viii) In the category of real (complex) $\mathcal{P}$-manifolds there exists a cross product. If $\mathcal{M} = (M, \mathcal{S})$ and $\mathcal{N} = (N, \mathcal{T})$ are two real (complex) $\mathcal{P}$-manifolds then their cross product is given by

$$\mathcal{M} \times \mathcal{N} := \left(M \times N, \mathrm{pr}_1^* \mathcal{S} \hat{\otimes} \mathrm{pr}_2^* \mathcal{T}\right) ,$$

and the canonical $\mathcal{P}$-projections by
$\mathrm{Pr}_1 := \left(\mathrm{pr}_1, (i_U)_{U \subset M \text{ open}}\right) : \mathcal{M} \times \mathcal{N} \to_{\mathcal{P}} \mathcal{M}$ and
$\mathrm{Pr}_2 := \left(\mathrm{pr}_2, (j_V)_{V \subset N \text{ open}}\right) : \mathcal{M} \times \mathcal{N} \to_{\mathcal{P}} \mathcal{N}$, where

$$i_U : \mathcal{S}(U) \hookrightarrow \left(\mathrm{pr}_1^* \mathcal{S} \hat{\otimes} \mathrm{pr}_2^* \mathcal{T}\right) \left(\mathrm{pr}_1^{-1}(U)\right) = \mathcal{S}(U) \hat{\otimes} \mathcal{T}(N)$$

and

$$j_V : \mathcal{T}(V) \hookrightarrow \left(\mathrm{pr}_1^* \mathcal{S} \hat{\otimes} \mathrm{pr}_2^* \mathcal{T}\right) \left(\mathrm{pr}_2^{-1}(V)\right) = \mathcal{S}(M) \hat{\otimes} \mathcal{T}(V) ,$$

$U \subset M$, $V \subset N$ open, denote the canonical inclusions.

By the universal property of the cross product there is a 1-1 correspondence between the $\mathcal{P}$-points $c \in_{\mathcal{P}} \mathcal{M} \times \mathcal{N}$ of $\mathcal{M} \times \mathcal{N}$ and pairs (a, b) of $\mathcal{P}$-points $a \in_P \mathcal{M}$ and $b \in_{\mathcal{P}} \mathcal{N}$ given by the assignment $c \mapsto (\mathrm{Pr}_1(c), \mathrm{Pr}_2(c))$.

Let me give two typical proofs:

First we prove the statement of (iii) in the real case (same proof in the complex case): "$\Rightarrow$" is trivial. "$\Leftarrow$" will be proven by induction on $n \in \mathbb{N}$ with $\mathcal{I}^n = 0$, $\mathcal{I} \lhd \mathcal{P}$ being the unique maximal ideal of $\mathcal{P}$. If $n = 1$ then of course the statement is trivial since then $\Phi_V(f) = f \circ \varphi|_{\varphi^{-1}(V)}$ for all $V \subset N$ open and $f \in \mathcal{C}^\infty(V)$.

Now let $\mathcal{I}^{n+1} = 0$. Then we define $\mathcal{Q} := \mathcal{P} / \mathcal{I}^n$. Clearly $\mathcal{Q}$ has $\mathcal{J} := \mathcal{I} / \mathcal{I}^n$ as unique maximal ideal, and $\mathcal{J}^n = 0$. Let $\natural : \mathcal{P} \to \mathcal{Q}$ denote the canonical projection. $\mathcal{M}^\natural := (M, \mathcal{S} / \mathcal{I}^n \mathcal{S})$ and $\mathcal{N}^\natural := (N, \mathcal{T} / \mathcal{I}^n \mathcal{T})$ are real $\mathcal{Q}$-manifolds, and $\natural$ induces sheaf projections

$$\natural : \mathcal{S} \to \mathcal{S} / \mathcal{I}^n \mathcal{S}$$

and

$$\natural : \mathcal{T} \to \mathcal{T} / \mathcal{I}^n \mathcal{T} .$$

Now let $\Phi = \left(\varphi, (\phi_V)_{V \subset N \text{ open}}\right)$ be a $\mathcal{P}$-morphism from $\mathcal{M}$ to $\mathcal{N}$. Then Φ induces a $\mathcal{Q}$-morphism $\Phi = \left(\varphi, \left(\phi_V^\natural\right)_{V \subset N \text{ open}}\right)$ from $\mathcal{M}^\natural$ to $\mathcal{N}^\natural$,

where for all $V \subset N$ open $\phi_V^\natural$ is the unique unital $\mathcal{P}$-algebra morphism $(\mathcal{T}/\mathcal{I}^n\mathcal{T})(V) \to (\mathcal{S}/\mathcal{I}^n\mathcal{S})\left(\varphi^{-1}(V)\right)$ such that

$$\begin{array}{ccc} \mathcal{T}(V) & \xrightarrow{\phi_V} & \mathcal{S}\left(\varphi^{-1}(V)\right) \\ \natural\downarrow & \circlearrowleft & \downarrow\natural \\ (\mathcal{T}/\mathcal{I}^n\mathcal{T})(V) & \underset{\phi_V^\natural}{\longrightarrow} & (\mathcal{S}/\mathcal{I}^n\mathcal{S})\left(\varphi^{-1}(V)\right) \end{array} .$$

Now we have to show that Φ is an isomorphism of ringed spaces. But since φ is already bijective it suffices to show that Φ is a local isomorphism. Therefore we may assume without loss of generality that $\mathcal{S} = \mathcal{C}_M^\infty \otimes \mathcal{P}$ and $\mathcal{T} = \mathcal{C}_N^\infty \otimes \mathcal{P}$. So we know that $\Phi_V(h) = h \circ \varphi|_{\varphi^{-1}(V)}$ for all $V \subset N$ open and $h \in \mathcal{C}^\infty(V) \otimes \mathcal{I}^n \hookrightarrow \mathcal{T}(V)$.

By induction hypothesis we already know that $\Phi^\natural$ is a $\mathcal{P}$-isomorphism. Let $V \subset N$ be open and $h \in \mathcal{S}\left(\varphi^{-1}(V)\right)$. Then $\phi_V^\natural$ is an isomorphism, and so there exists $f \in \mathcal{T}(V)$ such that $\phi_V^\natural\left(f^\natural\right) = h^\natural$, and therefore $\Delta := h - \phi_V(f) \in \mathcal{C}^\infty\left(\varphi^{-1}(V)\right) \otimes \mathcal{I}^n$. Since $\varphi : M \to N$ is a diffeomorphism we can build $\Delta \circ \varphi^{-1}\big|_V \in \mathcal{C}^\infty(V) \otimes \mathcal{I}^n$, and

$$\phi_V\left(f + \Delta \circ \varphi^{-1}\big|_V\right) = h - \Delta + \phi_V\left(\Delta \circ \varphi^{-1}\big|_V\right) = h$$

by (i) . This proves surjectivity of ϕ_V . For proving injectivity let $f \in \mathcal{T}(V)$ such that $\Phi_V(f) = 0$. Then $\Phi_V^\natural\left(f^\natural\right) = 0$, and so $f^\natural = 0$. Therefore $f \in \mathcal{C}^\infty(V) \otimes \mathcal{I}^n$, and so $0 = \Phi_V(f) = f \circ \varphi|_{\varphi^{-1}(V)}$. This implies $f = 0$. $\square$

Now we prove that $(N, \mathcal{T})$ in (vii) is indeed a $\mathcal{P}$-manifold of dimension $m - n$ in the real case (it is again the same proof in the complex case): Let $\mathbf{x}_0 \in N$. Then it is enough to show that there exists an open neighbourhood $U \subset M$ of $\mathbf{x}_0$ such that $(U \cap N, \mathcal{T}|_{U\cap N})$ is a real $\mathcal{P}$-manifold. So first of all choose a neighbourhood $U \subset M$ of $\mathbf{x}_0$ such that $(U \cap N, \mathcal{T}|_{U\cap N})$ is identified with an open subset of $\mathbb{R}^m$ regarded as real $\mathcal{P}$-manifold, and without loss of generality we may assume that $M = U$. So let Φ be given by the tuple $(f_1, \dots, f_n) \in (\mathcal{C}^\infty(U) \otimes \mathcal{P})^{\oplus n}$ and $\mathbf{a}$ by the tuple $(a_1, \dots, a_n) \in \left(\mathcal{P}^\mathbb{C}\right)^{\oplus n}$. Then after maybe replacing $(f_1, \dots, f_n)$ by $(f_1 - a_1, \dots, f_n - a_n)$ we may assume without loss of generality that $\mathbf{a} = \mathbf{0}$. Now let the $\mathcal{P}$-morphism $\widetilde{\Phi}$ from U to $\mathbb{R}^m$ be given by the tuple $(f_1, \dots, f_n, x_{n+1}, \dots, x_m)$, where $x_1, \dots, x_m \in \mathcal{C}^\infty(U)$ denote the standard coordinate functions on U . Then since $D\Phi^\#(\mathbf{x})$ is surjective at every point $\mathbf{x} \in U$ by assumption after maybe changing the order of the coordinates we assume without loss of generality that $D\widetilde{\Phi}^\#(\mathbf{x}_0) \in GL(m, \mathbb{R})$. So after maybe replacing U by a smaller open neighbourhood of $\mathbf{x}_0$ we may assume without loss of generality that $\varphi := \widetilde{\Phi}^\#$ is a diffeomorphism from U to $V := \varphi(U) \subset \mathbb{R}^m$, and so by (iii) $\widetilde{\Phi}$

is a $\mathcal{P}$-isomorphism from U to V . But then we see that $\Phi \circ \widetilde{\Phi}^{-1}$ is precisely the projection onto the first n coordinates, which is a usual smooth map from V to $\mathbb{R}^n$. So identifying U and V via $\widetilde{\Phi}$ we may without loss of generality assume that $\widetilde{\Phi} = \mathrm{id}$, and then the statement is trivial. $\Box$

Let us already here introduce the notion of $\mathcal{P}$- vector bundles over $\mathcal{P}$-manifolds. It will be crucial in section 5 .

Definition 1.3 ($\mathcal{P}$- vector bundles)

(i) Let $\mathcal{M} = (M, \mathcal{S})$ be a real (complex) $\mathcal{P}$-manifold. Then an $\mathcal{S}$- sheaf module $\mathcal{E}$ on M is called a real (holomorphic) $\mathcal{P}$- vector bundle of rank r over $\mathcal{M}$ iff it is locally isomorphic to $\mathcal{S}^{\oplus r}$. In this case $^\# : \mathcal{P} \to \mathbb{R}$ induces a body map $^\# : \mathcal{E} \to \Gamma^\infty(\diamondsuit, E)$ ($^\# : \mathcal{E} \to \Gamma^{\mathrm{hol}}(\diamondsuit, E)$), where $\Gamma^\infty(\diamondsuit, E)$ ($^\# : \mathcal{E} \to \Gamma^{\mathrm{hol}}(\diamondsuit, E)$) is the sheaf of smooth (holomorphic) sections of a real smooth (holomorphic) vector bundle $E \to M$ is of rank r , which is uniquely determined by $\mathcal{E}$. $\mathcal{E}^\# := E$ is called the body of $\mathcal{E}$. The space $H^0(\mathcal{E}) := \mathcal{E}(M)$ is called the space of global sections of $\mathcal{E}$, it is a $\mathcal{P}$- ($\mathcal{P}^{\mathbb{C}}$-) module. If $r = 1$ then $\mathcal{E}$ is called a $\mathcal{P}$- line bundle.

(ii) Let $\mathcal{E}$ be a real (holomorphic) $\mathcal{P}$- vector bundle over the real (complex) $\mathcal{P}$-manifold $\mathcal{M} = (M, \mathcal{S})$, and let $\Phi : \mathcal{N} \hookrightarrow \mathcal{M}$ be a $\mathcal{P}$-embedding of the real (holomorphic) $\mathcal{P}$-manifold $\mathcal{N} = (N, \mathcal{T})$.

$$\mathcal{E}|_{\mathcal{N}} := \mathcal{E}|_N \Big/ \mathfrak{m}\,\mathcal{E}|_N ,$$

where $\mathfrak{m}$ denotes the kernel of the canonical sheaf projection $\mathcal{S}|_N \to \mathcal{T}$, is called the restriction of the $\mathcal{P}$- vector bundle $\mathcal{E}$ to $\mathcal{N}$. It is a real (holomorphic) $\mathcal{P}$- vector bundle over $\mathcal{N}$ of rank r with body $E|_N$. If $U \subset M$ open and $F \in \mathcal{E}(U)$ then the image $F|_{\mathcal{N}}$ of F under the canonical map $\mathcal{E}(U) \to \mathcal{E}|_N\left(\varphi^{-1}(U)\right) \to \mathcal{E}|_{\mathcal{N}}\left(\varphi^{-1}(U)\right)$ is called the restriction of F to $\mathcal{N}$.

(iii) Let $\mathcal{E}$ and $\mathcal{F}$ be real (holomorphic) $\mathcal{P}$- vector bundles over the real (complex) $\mathcal{P}$-manifold $\mathcal{M} = (M, \mathcal{S})$ of rank r resp. s with bodies E resp. F . Then $\mathcal{E} \otimes \mathcal{F} := \mathcal{E} \otimes_{\mathcal{S}} \mathcal{F}$ is called the tensor product of $\mathcal{E}$ and $\mathcal{F}$. It is a $\mathcal{P}$- vector bundle over $\mathcal{M}$ of rank rs with body $E \otimes F$.

Let us collect some basic facts about $\mathcal{P}$- vector bundles:

(i) If E is a usual real smooth (holomorphic) vector bundle of rank r over the usual real smooth (complex) manifold M then E can be identified with the real (holomorphic) $\mathcal{P}$- vector bundle $\Gamma^\infty(\diamondsuit, E) \otimes \mathcal{P}$ ($\Gamma^{\mathrm{hol}}(\diamondsuit, E) \otimes \mathcal{P}^{\mathbb{C}}$) over M regarded as the real (complex) $\mathcal{P}$-manifold $(M, \mathcal{C}^\infty_M \otimes \mathcal{P})$ ($(M, \mathcal{O}_M \otimes \mathcal{P}^{\mathbb{C}})$) .

(ii) Let $\mathcal{E}$ be the real (holomorphic) $\mathcal{P}$- vector bundle of rank r over the real (holomorphic) $\mathcal{P}$-manifold $\mathcal{M} = (M, \mathcal{S})$. Then it admits local trivializations $\varphi_i : \mathcal{E}|_{U_i} \to (\mathcal{S}|_{U_i})^{\oplus r}$ being $\mathcal{S}|_{U_i}$-module isomorphisms for a suitable open cover $M = \bigcup_{i \in I} U_i$, $U_i \subset M$ open, $i \in I$, together with $\mathcal{P}$- transition functions $A_{ij} \in GL(r, \mathcal{S}(U_i \cap U_j))$ such that

$$\begin{array}{ccc} & \mathcal{E}|_{U_i \cap U_j} & \\ \varphi_i|_{U_i \cap U_j} \swarrow & \circlearrowleft & \searrow \varphi_j|_{U_i \cap U_j} \\ (\mathcal{S}|_{U_i \cap U_j})^{\oplus r} & \underset{A_{ij}}{\longrightarrow} & (\mathcal{S}|_{U_i \cap U_j})^{\oplus r} \end{array}$$

for all $i, j \in I$. $\mathcal{E}^{\#}$ then is given by local trivializations $U_i \times \mathbb{R}^r$ ($U_i \times \mathbb{C}^r$) , $i \in I$, together with the ordinary transition functions

$$A_{ij}^{\#} \in GL(r, \mathcal{C}^{\infty}(U_i \cap U_j)) \; (A_{ij}^{\#} \in GL(r, \mathcal{O}(U_i \cap U_j))) \, .$$

Again the kernel of $^{\#} : \mathcal{E} \to \Gamma^{\infty}(\diamondsuit, E)$ ($^{\#} : \mathcal{E} \to \Gamma^{\mathrm{hol}}(\diamondsuit, E)$) is $\mathcal{I}\mathcal{E}$ ($\mathcal{I}^{\mathbb{C}}\mathcal{E}$), and we have a canonical sheaf isomorphism
$\Gamma^{\infty}(\diamondsuit, E) \otimes \mathcal{I}^n \simeq \mathcal{I}^n \mathcal{E}$ ($\Gamma^{\mathrm{hol}}(\diamondsuit, E) \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^n \simeq \left(\mathcal{I}^{\mathbb{C}}\right)^n \mathcal{E}$) whenever $n \in \mathbb{N}$ such that $\mathcal{I}^{n+1} = 0$.

(iii) If in addition $\Phi = \left(\varphi, (\phi_U)_{U \subset M \text{ open}}\right) : \mathcal{N} \hookrightarrow_{\mathcal{P}} \mathcal{M}$ is a $\mathcal{P}$-imbedding of the real (holomorphic) $\mathcal{P}$-manifold $\mathcal{N} = (N, \mathcal{T})$ into $\mathcal{M}$ we get canonical maps $\mathcal{E}(U) \to \mathcal{E}|_N \left(\varphi^{-1}(U)\right) \to \mathcal{E}|_{\mathcal{N}} \left(\varphi^{-1}(U)\right)$ for all $U \subset M$ open, which are called the canonical restrictions, respecting the local trivializations $\left(\mathcal{T}|_{\varphi^{-1}(U_i)}\right)^r$ of $\mathcal{E}|_{\mathcal{N}}$ with the $\mathcal{P}$- transition functions

$$\phi_{U_i \cap U_j}(A_{ij}) \in GL\left(r, \mathcal{T}\left(\varphi^{-1}(U_i) \cap \varphi^{-1}(U_j)\right)\right) \, .$$

(iv) Now let the real (holomorphic) $\mathcal{P}$- vector bundles $\mathcal{E}$ and $\mathcal{F}$ over the real (complex) $\mathcal{P}$-manifold $\mathcal{M} = (M, \mathcal{S})$ be given by local trivializations $U_i \times \mathbb{R}^r$ ($U_i \times \mathbb{C}^r$) , $i \in I$, together with the $\mathcal{P}$- transition functions $A_{ij} \in GL(r, \mathcal{S}(U_i \cap U_j))$ resp. $B_{ij} \in GL(s, \mathcal{S}(U_i \cap U_j))$ then $\mathcal{E} \otimes \mathcal{F}$ is given by the $\mathcal{P}$- transition functions

$$A_{ij} \otimes B_{ij} \in GL(rs, \mathcal{S}(U_i \cap U_j)) \, .$$

Again if in addition $\Phi : \mathcal{N} \hookrightarrow_{\mathcal{P}} \mathcal{M}$ is a $\mathcal{P}$-imbedding of the real (holomorphic) $\mathcal{P}$-manifold $\mathcal{N} = (N, \mathcal{T})$ into $\mathcal{M}$ we see that
$(\mathcal{E} \otimes \mathcal{F})|_{\mathcal{N}} = \mathcal{E}|_{\mathcal{N}} \otimes \mathcal{F}|_{\mathcal{N}}$.

(v) One can show that if $\mathcal{E}$ is a real $\mathcal{P}$- vector bundle over the real smooth manifold M then $\mathcal{E} \simeq \mathcal{E}^{\#}$ as $\mathcal{P}$- vector bundles. This is again rigidity under local deformations.

Examples 1.4

Let $\mathcal{M} = (M, \mathcal{S})$ be a real (complex) $\mathcal{P}$-manifold of dimension n given by local $\mathcal{P}$-charts $U_i \subset \mathbb{R}^n$ ($U_i \subset \mathbb{C}^n$) open, $i \in I$, forming an open cover of M , and glueing data $\Phi_{ij} : U_i \underset{\text{open}}{\supset} U_{ij} \to_{\mathcal{P}} U_{ji} \underset{\text{open}}{\subset} U_j$. Then the tangent bundle $T\mathcal{M}$ and the cotangent bundle $T^*\mathcal{M}$ of $\mathcal{M}$ are the real (holomorphic) $\mathcal{P}$- vector bundles on $\mathcal{M}$ of rank n given by the local trivializations $(\mathcal{S}|_{U_i})^{\oplus n}$, $i \in I$, with transition functions $D\Phi_{ij}$ resp. $(D\Phi_{ij})^{-1}$, where the Jacobian $D\Phi_{ij} \in GL\left(n, \mathcal{S}\left(U_{ij}\right)\right)$ is taken componentwise from the tuple $(f_1, \dots, f_n) \in \left(\mathcal{S}\left(U_{ij}\right)\right)^{\oplus n}$ associated to Φ_{ij} , $i, j \in I$. So clearly $(T\mathcal{M})^\# = TM$ and $(T^*\mathcal{M})^\# = T^*M$.

Lemma 1.5 *Let $\mathcal{M}$ be a complex $\mathcal{P}$-manifold and $\mathcal{E}$ be a holomorphic $\mathcal{P}$-vector bundle over $\mathcal{M}$ with body $E \to M$. Let $d := \dim H^0(E) < \infty$. Then*

$$d \le \dim H^0(\mathcal{E}) \le d \dim \mathcal{P} \,,$$

and equivalent are

(i) $\dim H^0(\mathcal{E}) = d \dim \mathcal{P}$,

(ii) there exist $f_1, \dots, f_d \in H^0(\mathcal{E})$ such that $\left(f_1^\#, \dots, f_d^\#\right)$ is a basis of $H^0(E)$,

(iii) $H^0(\mathcal{E})$ is a free module over $\mathcal{P}^{\mathbb{C}}$ of rank d .

Furthermore if $f_1, \dots, f_d \in H^0(\mathcal{E})$ such that $\left(f_1^\#, \dots, f_d^\#\right)$ is a basis of $H^0(E)$ then $(f_1, \dots, f_d)$ is a $\mathcal{P}^{\mathbb{C}}$-basis of $H^0(\mathcal{E})$.

Proof: Let $\mathcal{I} \lhd \mathcal{P}$ denote the unique maximal ideal of $\mathcal{P}$. The first inequality is of course trivial if $\mathcal{I} = 0$. For $\mathcal{I} \neq 0$ let $N' \in \mathbb{N}$ be maximal such that $\mathcal{I}^{N'} \neq 0$. Then $H^0(E) \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^{N'} = \left(\mathcal{I}^{\mathbb{C}}\right)^{N'} H^0(\mathcal{E}) \sqsubset H^0(\mathcal{E})$, which proves the first inequality.

The second inequality, the implication (i) $\Rightarrow$ (ii) and the last statement will be proven by induction on $N \in \mathbb{N} \setminus \{0\}$ such that $\mathcal{I}^N = 0$. If $N = 1$ then $\mathcal{I} = 0$, and all statements are trivial.
Now assume $\mathcal{I}^{N+1} = 0$. Then again define $\mathcal{Q} := \mathcal{P} \big/ \mathcal{I}^N$ with unique maximal ideal $\mathcal{J} := \mathcal{I} \big/ \mathcal{I}^N$ having $\mathcal{J}^N = 0$, and let $\natural : \mathcal{P} \to \mathcal{Q}$ be the canonical projection. Let $\mathcal{E}^\natural := \mathcal{E} \Big/ \left(\mathcal{I}^{\mathbb{C}}\right)^N \mathcal{E}$, which is a holomorphic $\mathcal{Q}$-vector bundle over $\mathcal{M}^\natural$ of the same rank as $\mathcal{E}$, and let

$$\natural : H^0(\mathcal{E}) \to H^0\left(\mathcal{E}^\natural\right)$$

be the linear map induced by the canonical sheaf projection $\mathcal{E} \to \mathcal{E}^\natural$. Its kernel is $\left(\mathcal{I}^\mathbb{C}\right)^N H^0(\mathcal{E}) = H^0(E) \otimes \left(\mathcal{I}^\mathbb{C}\right)^N$. By induction hypothesis $\dim H^0\left(\mathcal{E}^\natural\right) \le d \dim \mathcal{Q}$, and so

$$\dim H^0(\mathcal{E}) \le d \dim \mathcal{Q} + d \dim \mathcal{I}^N = d \dim \mathcal{P} ,$$

which proves the second inequality.

For proving the implication (i) $\Rightarrow$ (ii) assume $\dim H^0(\mathcal{E}) = d \dim \mathcal{P}$. Then since $\dim \mathcal{P} = \dim \mathcal{Q} + \dim \mathcal{I}^N$, $\dim \left(H^0(E) \otimes \left(\mathcal{I}^\mathbb{C}\right)^N\right) = d \dim \mathcal{I}^N$ and $\dim H^0\left(\mathcal{E}^\natural\right) \le d \dim \mathcal{Q}$, we see that necessarily

$$\natural : H^0(\mathcal{E}) \to H^0\left(\mathcal{E}^\natural\right)$$

is surjective and $\dim H^0\left(\mathcal{E}^\natural\right) = d \dim \mathcal{Q}$. So by induction hypothesis and surjectivity there exist $f_1, \ldots, f_d \in H^0(\mathcal{E})$ such that $\left(f_1^\#, \ldots, f_d^\#\right)$ is a basis of $H^0(E)$, which proves (ii) .

For proving the last statement let $f_1, \ldots, f_d \in H^0(\mathcal{E})$ such that $\left(f_1^\#, \ldots, f_d^\#\right)$ is a basis of $H^0(E)$. Then by induction hypothesis $\left(f_1^\natural, \ldots, f_d^\natural\right)$ is a $\mathcal{Q}^\mathbb{C}$-basis of $H^0\left(\mathcal{E}^\natural\right)$. For proving that $(f_1, \ldots, f_d)$ spans $H^0(\mathcal{E})$ over $\mathcal{P}^\mathbb{C}$ let $F \in H^0(\mathcal{E})$. Then there exist $a_1, \ldots, a_d \in \mathcal{P}^\mathbb{C}$ such that

$$F^\natural = a_1^\natural f_1^\natural + \cdots + a_d^\natural f_d^\natural ,$$

and so

$$\Delta := F - a_1^\natural f_1^\natural - \cdots - a_d^\natural f_d^\natural \in \left(\mathcal{I}^\mathbb{C}\right)^N H^0(\mathcal{E}) = H^0(E) \otimes \left(\mathcal{I}^\mathbb{C}\right)^N .$$

Since $\left(f_1^\#, \ldots, f_d^\#\right)$ is a basis of $H^0(E)$ we see that there exist $b_1, \ldots, b_d \in \left(\mathcal{I}^\mathbb{C}\right)^N$ such that

$$\Delta = f_1^\# \otimes b_1 + \cdots + f_d^\# \otimes b_d = b_1 f_1 + \cdots + b_d f_d ,$$

and so

$$F = (a_1 + b_1) f_1 + \cdots + (a_d + b_d) f_d .$$

For proving linear independence let $a_1, \ldots, a_d \in \mathcal{P}^\mathbb{C}$ such that

$$a_1 f_1 + \cdots + a_d f_d = 0 .$$

Then $a_1^\natural f_1^\natural + \cdots + a_d^\natural f_d^\natural = 0$ in $H^0\left(\mathcal{E}^\natural\right)$, and so $a_1^\natural = \cdots = a_d^\natural = 0$. Therefore $a_1, \ldots, a_d \in \left(\mathcal{I}^\mathbb{C}\right)^N$, and this means

$$0 = a_1 f_1 + \cdots + a_d f_d = f_1^\# \otimes a_1 + \cdots + f_d^\# \otimes a_d \,.$$

Since $f_1^\#, \ldots, f_d^\#$ are linearly independent we get $a_1 = \cdots = a_d = 0$.

Now (ii) $\Rightarrow$ (iii) follows from the last statement, and (iii) $\Rightarrow$ (i) is of course trivial. $\square$

2 $\mathcal{P}$-lattices and automorphic forms

Let G be a real Lie group. Then it is in particular a smooth real manifold, and the multiplication on G can be written as a smooth map $m : G \times G \to G$. Therefore the multiplication turns the set $G^\mathcal{P}$ of all $\mathcal{P}$-points of G into a group via $gh := m(g,h)$ for all $g, h \in_\mathcal{P} G$, and clearly $^\# : G^\mathcal{P} \to G$, $g \mapsto g^\#$ is a group epimorphism. Of course the $\mathcal{P}$-points of $GL(n, \mathbb{R})$ are in 1-1 correspondence with $n \times n$ -matrices having entries in $\mathcal{P}$ and body (taken componentwise) in $GL(n, \mathbb{R})$, and the product of two $\mathcal{P}$-points of $GL(n, \mathbb{R})$ can be computed via ordinary matrix multiplication.

Definition 2.1 ($\mathcal{P}$-lattices) *Let Υ be a subgroup of $G^\mathcal{P}$. Υ is called a $\mathcal{P}$-lattice of G iff*

{*i*} $\Upsilon^\# := \left\{ \gamma^\# \,\middle|\, \gamma \in \Upsilon \right\} \sqsubset G$ *is an ordinary lattice, called the body of* Υ *, and*

{*ii*} $^\# : \Upsilon \to \Upsilon^\#$, $\gamma \mapsto \gamma^\#$ *is bijective and so automatically an isomorphism.*

Obviously a $\mathcal{P}$-lattice Υ is nothing but a local deformation over the algebra $\mathcal{P}$ of the natural embedding $\Gamma := \Upsilon^\# \hookrightarrow G$ as a group homomorphism. Of course given a $\mathcal{P}$-lattice Υ of G with body $\Gamma \sqsubset G$ and $g \in_\mathcal{P} G$ with $g^\# = 1$ we get another $\mathcal{P}$-lattice $g \Upsilon g^{-1}$ of G with same body Γ . The set of all $\mathcal{P}$-lattices of G of the form $g \Upsilon g^{-1}$ of G , $g \in_\mathcal{P} G$, $g^\# = 1$, is called the conjugacy class of Υ . In the case $\mathcal{I}^2 = 0$, where $\mathcal{I} \lhd \mathcal{P}$ denotes the unique maximal ideal of $\mathcal{P}$, given an ordinary lattice Γ the conjugacy classes of $\mathcal{P}$-lattices Υ with body Γ are in 1-1 correspondence with $H^1(\Gamma, \mathfrak{g}) \otimes \mathcal{I}$, Γ acting on the Lie algebra $\mathfrak{g}$ of G by Ad , see for example [8] .

Lemma 2.2

(i) Let Υ be a $\mathcal{P}$-lattice of G , $\gamma \in_\mathcal{P} \Upsilon$ and $n \in \mathbb{N} \setminus \{0\}$ such that $\left(\gamma^\#\right)^n = 1$. Then $\gamma^n = 1$.

(ii) Let $g \in_\mathcal{P} G$ with $g^n = 1$ and $g^\# \in Z(G)$. Then $g = g^\#$.

Proof: (i) Obviously $\gamma^n \in \Upsilon$ having $(\gamma^n)^\# = 1$. Therefore $\gamma^n = 1$ by property {ii} . $\square$

(ii) Clearly $\left(g^\#\right)^n = (g^n)^\# = 1$. And so it suffices to show that $\varphi : G \to G$, $h \mapsto h^n$ is a local diffeomorphism at $g^\#$. But, since $g^\# \in Z(G)$ one can easily compute

$$D\varphi\left(g^\#\right) = n\,\left(Dt_{g^\#}(1)\right)^{-1} ,$$

which is invertible since the translation $t_{g^\#} : G \to G$ with $g^\#$ is a diffeomorphism. $\square$

Let $H := \{z \in \mathbb{C} \,|\, \mathrm{Im}\ z > 0\}$ be the usual upper half plane, and from now on let $G := SL(2, \mathbb{R})$. Then G acts on H via Möbius transformations

$$gz := \frac{az+b}{cz+d} \, , \, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

more precisely we have a group epimorphism $\bar{\ } : G \to \mathrm{Aut}(H)$ with kernel $\{\pm 1\} = Z(G)$. The action of G on H induces a morphism of ringed spaces

$$\left(G \times H, \mathrm{pr}_1^* \left(\mathcal{C}_G^\infty\right)^{\mathbb{C}} \hat{\otimes} \mathrm{pr}_2^* \mathcal{O}_H\right) \to (H, \mathcal{O}_H) ,$$

and therefore a group homomorphism $G^{\mathcal{P}} \to \{\mathcal{P}\text{-automorphisms of } H\}$ respecting $^\#$ with kernel $\{\pm 1\}$, which is no longer surjective if $\mathcal{P} \neq \mathbb{R}$. If $g \in_{\mathcal{P}} G$ then g as a $\mathcal{P}$-automorphism of H is given by $gz \in \mathcal{O}(H) \otimes \mathcal{P}^{\mathbb{C}}$. For all $U \subset H$ open and $f \in \mathcal{O}(U) \otimes \mathcal{P}^{\mathbb{C}}$ denote by $f(gz) \in \mathcal{O}\left(g^{-1}U\right) \otimes \mathcal{P}^{\mathbb{C}}$ the pullback of f under g . If U is invariant under $g^\#$ then we say f is g-invariant if and only if $f(gz) = f$.

Let $k \in \mathbb{N}$ be fixed for the rest of the article, and let $j \in \mathcal{C}^\infty(G)^{\mathbb{C}} \hat{\otimes} \mathcal{O}(B)$ be given by

$$j(g,z) := \frac{1}{cz+d} \, , \, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Then j fulfills the cocycle property $j(gh,z) = j(g,hz)j(h,z)$, and an easy computation shows that $j(g,z)^2 = g'(z)$, g regarded as an automorphism of H .

This gives a right-action of G on $\mathcal{O}(H)$ by

$$| : \mathcal{O}(H) \to \mathcal{C}^\infty(G)^{\mathbb{C}} \hat{\otimes} \mathcal{O}(H) \, , \, f|_g(z) := f(gz)j(g,z)^k ,$$

and this induces an action of $G^{\mathcal{P}}$ on $\mathcal{O}(H) \otimes \mathcal{P}^{\mathbb{C}}$ given by

$$|_g : \mathcal{O}(H) \otimes \mathcal{P}^{\mathbb{C}} \to \mathcal{O}(H) \otimes \mathcal{P}^{\mathbb{C}} \, , \; f|_g(z) := f(gz) j(g,z)^k$$

for all $g \in_{\mathcal{P}} G$, or more precisely for all $U \subset H$ this gives a $\mathcal{P}^{\mathbb{C}}$-linear map

$$|_g : \mathcal{O}(U) \otimes \mathcal{P}^{\mathbb{C}} \to \mathcal{O}\left(g^{-1}U\right) \otimes \mathcal{P}^{\mathbb{C}} \, , \; f|_g(z) := f(gz) j(g,z)^k \, .$$

From now on let Υ be a fixed $\mathcal{P}$-lattice in G with body $\Gamma := \Upsilon^{\#}$.

Examples 2.3

(i) Let $\Gamma := SL(2,\mathbb{Z})$. Then Γ is the free group generated by the two matrices $R := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ and $S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ modulo the relations $R^3 = S^4 = 1$. One can easily compute that the equations $g^3 = h^4 = 1$ define a smooth submanifold M of G^2 of dimension 4 near the point (R,S) and that the map

$$\varphi : G \to M \, , \; g \mapsto \left(g^{-1}Rg, g^{-1}Sg\right)$$

has injective differential at $g = 1$. So take any smooth submanifold $M' \subset M$ of dimension 1 such that $T_{(R,S)}M = T_{(R,S)}M' \oplus \operatorname{Im} D\varphi(1)$. Then obviously the conjugacy classes of $\mathcal{P}$-lattices Υ with $\Upsilon^{\#} = \Gamma$ are in 1-1 correspondence with $\mathcal{P}$-points $x \in_{\mathcal{P}} M'$ having $x^{\#} = (R,S)$, and so via a local chart of M' at (R,S) with $\mathcal{I}$. In particular

$$H^1(\Gamma, \mathfrak{g}) = T_{(R,S)}M \big/ \operatorname{Im} D\varphi(1) \simeq \mathfrak{g} \big/ (\mathfrak{z}_{\mathfrak{g}}(R) + \mathfrak{z}_{\mathfrak{g}}(S))$$

has dimension 1 .

(ii) Let X be a compact RIEMANN surface of genus g , $s_1, \dots, s_m \in X$, $3g + m \geq 3$. Then the universal covering of $X \setminus \{s_1, \dots, s_m\}$ is isomorphic to H , and by [7] one can write $X \setminus \{s_1, \dots, s_m\} = \Gamma \setminus H$, where $\Gamma \subset G$ is a lattice without elliptic elements having $-1 \notin \Gamma$, it is the free group generated by some hyperbolic elements $A_1, B_1, \dots, A_g, B_g \in G$ and parabolic elements $C_1, \dots, C_n \in G$ modulo the single relation

$$[A_1, B_1] \dots [A_g, B_g] \, C_1 \dots C_m = 1 \, .$$

Then by exactly the same method as in (i) one obtains a 1-1 correspondence between the conjugacy classes of $\mathcal{P}$-lattices Υ with $\Upsilon^{\#} = \Gamma$

and $\mathcal{P}$-points x of a suitable $(6(g-1)+3m)$-dimensional smooth submanifold of G^{2g+m} having $x^{\#} = (A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_m)$ and so with $\mathcal{I}^{\oplus(6(g-1)+3m)}$. In particular

$$\dim H^1(\Gamma, \mathfrak{g}) = 6(g-1) + 3m\,.$$

For defining automorphic forms with respect to the $\mathcal{P}$-lattice Υ we need some more informtion about the behaviour of Γ being an ordinary lattice in G . Let $\mathbb{P}^1$ denote the RIEMANN sphere, on which $SL(2,\mathbb{C})$ acts via MÖBIUS transformations. For $z \in H \cup \partial_{\mathbf{P}^1} H$ denote by $\overline{z} := \Gamma z$ the image of z under the canonical projection $H \to \Gamma\backslash H$ resp. $\partial_{\mathbf{P}^1} H \to \Gamma\backslash\partial_{\mathbf{P}^1} H$.

Definition 2.4

(i) An element $\overline{z_0} \in \Gamma\backslash H$, $z_0 \in H$, *is called regular iff* $\overline{\Gamma}^{z_0} = \{\mathrm{id}\}$,

(ii) an element $\overline{z_0} \in \Gamma\backslash H$, $z_0 \in H$, *is called elliptic iff* $\overline{\Gamma}^{z_0} \neq \{\mathrm{id}\}$, *and finally*

(iii) an element $\overline{z_0} \in \Gamma\backslash\partial_{\mathbf{P}^1} H$, $z_0 \in \partial_{\mathbf{P}^1} H$, *is called a cusp of* $\Gamma\backslash H$ *iff* $\overline{\Gamma} \cap \overline{P^{z_0}} \neq \{\mathrm{id}\}$, *where* $P^{z_0} \sqsubset G$ *denotes the parabolic subgroup associated to* z_0 .

It is a well known fact that there exist always only finitely many elliptic points in $\Gamma\backslash H$, $\Gamma\backslash H$ has always only finitely many cusps, and the quotient $\Gamma\backslash H$ can be compactified as a topological space by adding the cusps of $\Gamma\backslash H$. This can for example be deduced from theorem 0.6 in [3] .

Since G acts transitively on H we see that for each $z_0 \in H$ there exists $g \in G$ such that $gi = z_0$, and therefore $G^{z_0} = gKg^{-1}$, where $K := G^i \simeq \mathbb{R}/\mathbb{Z}$ is a maximal compact subgroup of G . Therefore if z_0 is an elliptic point of $\Gamma\backslash H$ then $\Gamma^{z_0} \sqsubset G^{z_0}$ and $\overline{\Gamma}^{z_0} \sqsubset \overline{G}^{z_0}$ are finite non-trivial cyclic groups. $\mathrm{ord}\,\overline{\Gamma}^{z_0} \in \mathbb{N}$ is called the period of $\overline{z_0}$.

Since furthermore G acts transitively on the boundary $\partial_{\mathbf{P}^1} H$ of H we see that for each $z_0 \in \partial_{\mathbf{P}^1} H$ there exists an element $g \in G$ such that $g(\infty) = z_0$, and so $P^{z_0} = gP^{\infty}g^{-1}$. Recall that $P^{\infty} \simeq \mathbb{R}$ is the one-parameter-subgroup generated by $\chi_0 \in \mathfrak{g}$, $\mathfrak{g}$ being the LIE algebra of G , with

$$\chi_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Therefore if z_0 is a cusp of $\Gamma\backslash H$ then $\overline{\Gamma} \cap \overline{P^{z_0}}$ is infinite cyclic, and one can always choose $g \in G$ such that in addition $g^{-1}\overline{\Gamma}g \cap \overline{P^{\infty}} = \langle \overline{g_0} \rangle$, where

$$g_0 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \exp(\chi_0) \ .$$

Lemma 2.5

(i) Let $g \in_{\mathcal{P}} G$ *such that* $g^{\#} = g_0$ *. Then there exists a unique* $\chi \in \mathfrak{g} \otimes \mathcal{P}$ *such that* $\chi^{\#} = \chi_0$ *and* $g = \exp(\chi)$ *.*

(ii) Let $\chi \in \mathfrak{g} \otimes \mathcal{P}$ *with body* $\chi^{\#} = \chi_0$ *. Then there exists a* $\mathcal{P}$*-automorphism* $\Omega : H \to_{\mathcal{P}} H$ *such that* $\Omega^{\#} = \mathrm{id}$ *and*

$$\begin{array}{rccl} & H & \xrightarrow{\Omega} & H \\ \exp(t\chi_0) : z \mapsto z + t \downarrow & & \circlearrowleft & \downarrow \exp(t\chi) \\ & H & \xrightarrow[\Omega]{} & H \end{array}$$

for all $t \in \mathbb{R}$ *. All other* $\mathcal{P}$*-automorphisms with this property are given by* $z \mapsto \Omega(z + a)$ *where* $a \in \mathcal{I}^{\mathbb{C}}$ *.*

Proof: (i) For proving this statement it suffices to show that exp is a local diffeomorphism at χ_0 . We use theorem 1.7 of chapter II section 1.4 in [6] , which says the following:

> Let G be a Lie group with Lie algebra $\mathfrak{g}$. The exponential mapping of the manifold $\mathfrak{g}$ into G has the differential
>
> $$D\exp_X = D\left(l_{\exp X}\right)_e \circ \frac{1 - e^{-\mathrm{ad}_X}}{\mathrm{ad}_X} \qquad (X \in \mathfrak{g}) \,.$$
>
> As usual, $\mathfrak{g}$ is here identified with the tangent space $\mathfrak{g}_X$.

Hereby e denotes the unit element of the Lie group G , l_g denotes the left translation on G with an element $g \in G$, and exp is used as a local chart of G at e .

Since χ_0 is nilpotent in $\mathfrak{g}$ we see that also $\mathrm{ad}_{\chi_0} \in \mathrm{End}(\mathfrak{g})$ is nilpotent, and so is $\frac{1-e^{-\mathrm{ad}_{\chi_0}}}{\mathrm{ad}_{\chi_0}} - 1 \in \mathrm{End}(\mathfrak{g})$. Since $l_{\exp \chi_0} : G \to G$ is a diffeomorphism, we obtain the desired result applying the theorem with $X := \chi_0$. $\Box$

(ii) Since $\chi^{\#} = \chi_0$ an easy calculation shows that χ is nilpotent as a matrix with entries in $\mathcal{P}$. Therefore since in addition $\exp(t\chi_0)$ is upper triangle we see that $\omega := \exp(t\chi)\, i \in \mathcal{P}^{\mathbb{C}}[t]$ with body $\omega^{\#} = \exp\left(t\chi^{\#}\right)\, i = t + i$. Now let $\Omega := \omega(t - i) \in \mathcal{P}^{\mathbb{C}}[t]$. Then $\Omega^{\#} = t$, and so Ω can be regarded as a $\mathcal{P}$-automorphism $\Omega : H \to H$ having $\Omega^{\#} = \mathrm{id}$.

Since everything in the diagramme is given by tuples of holomorphic functions on H it suffices to prove its commutativity on the non discrete subset $\mathbb{R}+i \subset H$. So let $t, u \in \mathbb{R}$. Then

$$\begin{aligned}(\Omega \circ \exp(t\chi_0))(u+i) &= \Omega(u+i+t) \\ &= \omega(u+t) \\ &= \exp(t\chi)\exp(u\chi)i \\ &= \exp(t\chi)\omega(u) \\ &= (\exp(t\chi) \circ \Omega)(u+i)\,.\end{aligned}$$

Now let $\widetilde{\Omega} : H \to_{\mathcal{P}} H$ be another $\mathcal{P}$-automorphism. Then $\widetilde{\Omega}$ has the same properties iff $\Omega^{-1} \circ \widetilde{\Omega}$ is a $\mathcal{P}$-automorphism with body id and commuting with all translations $H \to H\,, z \mapsto z+t$, $t \in \mathbb{R}$, iff $\left(\Omega^{-1} \circ \widetilde{\Omega}\right)(z) = z + a$ with some $a \in \mathcal{O}(H) \otimes \mathcal{I}^{\mathbb{C}}$ and invariant under the translations $H \to H$, $z \mapsto z+t$, $t \in \mathbb{R}$, and therefore constant. $\square$

Definition 2.6 *Let* $z_0 \in \partial_{\mathbf{P}^1} H$, $\gamma \in_{\mathcal{P}} G$ *such that* $\gamma^\# \in P^{z_0} \setminus \{\pm 1\}$, $U \subset H$ *open and* $\gamma^\#$*-invariant and finally* $f \in \mathcal{O}(U) \otimes \mathcal{P}^{\mathbb{C}}$ *such that* $f|_\gamma = f$ *or* f γ*-invariant (which is nothing but* $f|_\gamma = f$ *for* $k=0$ *). Let* $g \in G$ *such that* $g\infty = z_0$ *and* $g_0 = g^{-1}\gamma^\# g$ *. Furthermore assume that there exists* $c > 0$ *such that*

$$\{\mathrm{Im}\ z > c\} \subset g^{-1}U\,,$$

which is g_0*-invariant. Let* χ *and* Ω *be given by lemma 2.5 taken* $\widetilde{g_0} := g^{-1}\gamma^\# g$ *instead of* g *having body* g_0 *. If we define* $f|_g|_\Omega$ *as*

$$f|_g|_\Omega\,(z) := f|_g\,(\Omega z)\,\Omega'(z)^{\frac{k}{2}}$$

then we see that

$$f|_g|_\Omega\,(z) = \left.f|_g|_\Omega\right|_{g_0}(z) = f|_g|_\Omega\,(z+1)\,.$$

Now f *is called bounded (vanishing) at* z_0 *iff* $f|_g|_\Omega\,(z)$ *is bounded, and therefore converging, (resp. vanishing) for* $\mathrm{Im}\ z \rightsquigarrow \infty$ *.*

Observe that $(\Omega')^r$ is well defined for all $r \in \mathbb{R}$ since $\Omega^\# = \mathrm{id}$, and so $(\Omega')^\# = 1$. Clearly the definition does not depend on the choice of g and Ω because g is uniquely determined up to ± 1 , and if $\widetilde{\Omega}$ is another choice for Ω then $\widetilde{\Omega}(z) = \Omega(z+a)$ by lemma 2.5 with some $a \in \mathcal{I}^{\mathbb{C}}$. Therefore $f|_g|_{\widetilde{\Omega}}\,(z) = f|_g|_\Omega\,(z+a)$.

Definition 2.7 (automorphic and cusp forms for Υ) *Let $f \in \mathcal{O}(H) \otimes \mathcal{P}^{\mathbb{C}}$. f is called an automorphic (cusp) form for Υ of weight k iff*

(i) $f|_\gamma = f$ *for all* $\gamma \in_{\mathcal{P}} \Upsilon$,

(ii) f *is bounded (vanishing) at all cusps* $\overline{z_0} \in \Gamma \backslash \partial_{\mathbf{P}^1} H$ *of* $\Gamma \backslash H$.

The space of automorphic (cusp) forms for Υ of weight k is denoted by $M_k(\Upsilon)$ (resp. $S_k(\Upsilon)$). We have $S_k(\Upsilon) \sqsubset M_k(\Upsilon) \sqsubset \mathcal{O}(H) \otimes \mathcal{P}^{\mathbb{C}}$ as $\mathcal{P}^{\mathbb{C}}$-submodules.

Since $(f|_g)^\# = f^\#\big|_{g^\#}$ for all $f \in \mathcal{O}(H) \otimes \mathcal{P}^{\mathbb{C}}$ and $g \in_{\mathcal{P}} G$ we see that $M_k(\Upsilon)^\# = M_k(\Gamma)$ and $S_k(\Upsilon)^\# = S_k(\Gamma)$. Using lemma 2.2 we observe that $-1 \in \Gamma \Leftrightarrow -1 \in \Upsilon$, and so in this case $M_k(\Upsilon) = 0$ if $2 \not| k$.

Theorem 2.8

$$\dim M_k(\Gamma) \leq \dim M_k(\Upsilon) \leq \dim M_k(\Upsilon) \dim \mathcal{P}$$
$$(\dim S_k(\Gamma) \leq \dim S_k(\Upsilon) \leq \dim S_k(\Upsilon) \dim \mathcal{P}) ,$$

and equivalent are

(i) $\dim M_k(\Upsilon) = \dim M_k(\Gamma) \dim \mathcal{P}$ *(* $\dim S_k(\Upsilon) = \dim S_k(\Gamma) \dim \mathcal{P}$ *),*

(ii) there exist $f_1, \dots, f_r \in M_k(\Upsilon)$ ($f_1, \dots, f_r \in S_k(\Upsilon)$) such that $\left(f_1^\#, \dots, f_r^\#\right)$ is a basis of $M_k(\Gamma)$ ($S_k(\Gamma)$),

(iii) $M_k(\Upsilon)$ ($S_k(\Upsilon)$) is a free module over $\mathcal{P}^{\mathbb{C}}$ of rank $\dim M_k(\Gamma)$ *(* $\dim S_k(\Gamma)$ *).*

Furthermore if $f_1, \dots, f_r \in M_k(\Upsilon)$ ($f_1, \dots, f_r \in S_k(\Upsilon)$) such that $\left(f_1^\#, \dots, f_r^\#\right)$ is a basis of $M_k(\Gamma)$ ($S_k(\Gamma)$) then $(f_1, \dots, f_r)$ is a $\mathcal{P}^{\mathbb{C}}$-basis of $M_k(\Upsilon)$ ($S_k(\Upsilon)$).

Proof: This is a corollary of lemma 1.5 since in section 3 and 5 we will show that $M_k(\Gamma)$ and $S_k(\Gamma)$ are the spaces of global sections for certain holomorphic $\mathcal{P}$- line bundles on $\Upsilon \backslash H \cup \{$ cusps of $\Gamma \backslash H\}$ as $\mathcal{P}$- Riemann surface. $\square$

Aim of this article is now to prove that for large weights k we have an isomorphism $M_k(\Upsilon) \simeq M_k(\Gamma) \otimes \mathcal{P}^{\mathbb{C}}$ mapping $S_k(\Upsilon)$ to $S_k(\Gamma) \otimes \mathcal{P}^{\mathbb{C}}$.

3 The quotient as a $\mathcal{P}$- Riemann surface

It is a well known fact that there exists a structure of a compact Riemann surface on $X := \Gamma\backslash H \cup \{$ cusps of $\Gamma\backslash H\}$ such that the subsheaf of $\mathcal{O}_H$ of Γ-invariant functions is the pullback of $\mathcal{O}_X|_{\Gamma\backslash H}$ under the canonical projection $\pi : H \to \Gamma\backslash H \hookrightarrow X$. Now we will construct a $\mathcal{P}$- Riemann surface $\mathcal{X} = (X, \mathcal{S})$ such that the subsheaf of $\mathcal{O}_H \otimes \mathcal{P}^{\mathbb{C}}$ of Υ-invariant functions is precisely the pullback of $\mathcal{S}|_{\Gamma\backslash H}$ under the canonical projection $\pi : H \to \Gamma\backslash H$. For this purpose define the sheaf $\mathcal{S}$ of $\mathcal{P}^{\mathbb{C}}$-algebras on X as

$$\mathcal{S}(V) := \Big\{ f \in \mathcal{O}\left(\pi^{-1}(V)\right) \otimes \mathcal{P}^{\mathbb{C}} \quad \Upsilon\text{-invariant and bounded at all cusps } \overline{z_0} \in V \text{ of } \Gamma\backslash H \Big\}$$

for all $V \subset X$ open. Recall that a function $f \in \mathcal{O}\left(\pi^{-1}(V)\right) \otimes \mathcal{P}^{\mathbb{C}}$ is called Υ-invariant iff $f(\gamma z) = f$ for all $\gamma \in \Upsilon$. Now one has to show that locally $\mathcal{S} \simeq \mathcal{O}_X \otimes \mathcal{P}^{\mathbb{C}}$. We will do this giving local $\mathcal{P}$-charts for $\mathcal{X}$. On may define the $\mathcal{P}$- Riemann surface $\Upsilon\backslash H := \left(\Gamma\backslash H, \mathcal{S}|_{\Gamma\backslash H}\right)$ to be the quotient of H by Υ and the $\mathcal{P}$-morphism $\Pi := \left(\pi, (i_V)_{V \subset \Gamma\backslash H \text{ open}}\right)$ from H to $\Upsilon\backslash H$ as the canonical $\mathcal{P}$-projection, where $i_V : \mathcal{S}(V) \hookrightarrow \mathcal{O}\left(\pi^{-1}(V)\right) \otimes \mathcal{P}^{\mathbb{C}}$, $V \subset \Gamma\backslash H$ open, denote the canonical inclusions.

Let $\mathcal{F}$ be the $\mathcal{S}$- sheaf module on X given by

$$\mathcal{F}(V) := \Big\{ f \in \mathcal{O}\left(\pi^{-1}(V)\right) \otimes \mathcal{P}^{\mathbb{C}} \,\Big|\, f|_\gamma = f \text{ for all } \gamma \in \Upsilon \text{ and } f \text{ bounded at all cusps } \overline{z_0} \in V \text{ of } \Gamma\backslash H \Big\}$$

for all $V \subset X$ open. If $-1 \in \Gamma$ and k is odd then of course $\mathcal{F} = 0$, and in the cases where either k is even or k is odd and $-1 \notin \Gamma$ we will show that $\mathcal{F}$ is a holomorphic $\mathcal{P}$- line bundle over $\mathcal{X}$. For this purpose we have to give local trivializations $\mathcal{F} \simeq \mathcal{S}$ as $\mathcal{S}$- sheaf modules. Then obviously $M_k(\Upsilon) = H^0(\mathcal{F})$.

For $g \in G$ let $\pi_g : H \to \langle g \rangle \backslash H$ denote the canonical projection.

At regular points of $\Gamma\backslash H$:

> Let $\overline{z_0} \in \Gamma\backslash H$, $z_0 \in H$, be regular. Then there exists an open neighbourhood $U \subset H$ of z_0 auch that $\gamma U \cap U = \emptyset$ for all $\gamma \in \overline{\Gamma} \setminus \{1\}$, and so

$$\pi|_U : U \to \pi(U) \underset{\text{open}}{\subset} X$$

is biholomorphic. Its inverse is a local chart of X at $\overline{z_0}$.

For giving a local $\mathcal{P}$-chart of $\mathcal{X}$ at $\overline{z_0}$ we will show that locally Π is an isomorphism at z_0 . Indeed the restriction of Π to U is given by $\Pi|_U = \left(\pi|_U, \left(|_{\pi^{-1}(V)\cap U}\right)_{V\subset\pi(U) \text{ open}}\right)$ from $\left(U, \mathcal{O}_U \otimes \mathcal{P}^{\mathbb{C}}\right)$ to $\left(\pi(U), \mathcal{S}|_{\pi(U)}\right)$ as ringed spaces, where for all $V \in \pi(U)$ open

$$\begin{aligned} |_{\pi^{-1}(V)\cap U} : \mathcal{S}(V) = \left\{ f \in \mathcal{O}\left(\pi^{-1}(V)\right) \otimes \mathcal{P}^{\mathbb{C}} \quad \Upsilon\text{-invariant} \right\} \\ \to \mathcal{O}\left(\pi^{-1}(V) \cap U\right) \otimes \mathcal{P}^{\mathbb{C}} \end{aligned}$$

simply denotes the restriction map. It is indeed an isomorphism of $\mathcal{P}^{\mathbb{C}}$-algebras since $\pi^{-1}(V) = \dot{\bigcup}_{\gamma\in\overline{\Gamma}} \gamma\left(\pi^{-1}(V) \cap U\right)$ for all $V \subset \pi(U)$ open. So $\left(\Pi|_U\right)^{-1}$ is a local $\mathcal{P}$-chart for $\mathcal{X}$.

For giving a local trivialization of $\mathcal{F}$ at z_0 identify the ringed spaces $\left(U, \mathcal{O}_U \otimes \mathcal{P}^{\mathbb{C}}\right)$ and $\left(\pi(U), \mathcal{S}|_{\pi(U)}\right)$ via $\Pi|_U$. Then we see by the same argument that the restriction maps

$$|_{\pi^{-1}(V)\cap U} : \mathcal{F}(V) \to \mathcal{O}\left(\pi^{-1}(V) \cap U\right) \otimes \mathcal{P}^{\mathbb{C}} ,$$

$V \subset \pi(U)$ open, give an isomorphism of the $\mathcal{S}|_{\pi(U)}$- sheaf modules $\mathcal{F}|_{\pi(U)}$ and $\mathcal{O}_U \otimes \mathcal{P}^{\mathbb{C}}$.

At elliptic points of $\Gamma\backslash H$:

Let $\overline{z_0} \in \Gamma\backslash H$, $z_0 \in H$, be elliptic of period n and $g \in G$ such that $gi = z_0$. Then $\Gamma^{z_0} = \langle\gamma_0\rangle \subset G^{z_0} = gKg^{-1}$ for some $\gamma_0 \in \Gamma^{z_0}$.

Let $\overline{\pi} : \langle\gamma_0\rangle \backslash H \to \Gamma\backslash H$ denote the canonical projection. Then $\pi = \overline{\pi} \circ \pi_{\gamma_0}$. Now we choose

$$c : B := \left\{ z \in \mathbb{C} \,\middle|\, |z| < 1 \right\} \to H \, , \, z \mapsto -i\,\frac{z+1}{z-1}$$

as a Cayley transform with $c(0) = i$. It is clearly biholomorphic, and $c^{-1}g^{-1}\gamma_0 gc$ fixes 0 as an automorphism of B and so is given by multiplication with a suitable $\eta \in U(1)$ of order n . Therefore

$$\varphi : B \xrightarrow{\sqrt[n]{}} \langle\eta\rangle\backslash B \xrightarrow{c} \left\langle g^{-1}\gamma_0 g\right\rangle\backslash H \xrightarrow{g} \langle\gamma_0\rangle\backslash H \xrightarrow{\overline{\pi}} \Gamma\backslash H$$

gives a locally biholomorphic map at $0 \mapsto 0 \mapsto i \mapsto z_0 \mapsto \overline{z_0}$, and so φ^{-1} is a local chart of $\Gamma\backslash H$ at $\overline{z_0}$.

For giving a local $\mathcal{P}$-chart of $\mathcal{X}$ at z_0 define the sheaves $\widetilde{\mathcal{O}}_B$, $\widetilde{\mathcal{O}}_H$ of unital complex algebras and the sheaf $\widetilde{\mathcal{S}}$ of unital $\mathcal{P}^{\mathbb{C}}$-algebras on $\langle\eta\rangle\backslash B$, $\left\langle g^{-1}\gamma_0 g\right\rangle\backslash H$ resp. $\langle\gamma_0\rangle\backslash H$ by

$$\widetilde{\mathcal{O}}_B(V) := \left\{ f \in \mathcal{O}\left(\pi_\eta^{-1}(V)\right) \,\middle|\, f(\eta w) = f \right\} ,$$

$$\widetilde{\mathcal{O}}_H(V) := \left\{ f \in \mathcal{O}\left(\pi_{g^{-1}\gamma_0 g}^{-1}(U)\right) \,\middle|\, f\left(g^{-1}\gamma_0 g z\right) = f \right\}$$

and

$$\widetilde{\mathcal{S}}(V) := \left\{ f \in \mathcal{O}\left(\pi_{\gamma_0}^{-1}(V)\right) \otimes \mathcal{P}^{\mathbb{C}} \,\middle|\, f\left(\widetilde{\gamma_0} z\right) = f \right\}$$

for all $V \subset \langle\eta\rangle\backslash B$, $V \subset \left\langle g^{-1}\gamma_0 g\right\rangle\backslash H$ resp. $V \subset \langle\gamma_0\rangle\backslash H$, where $\pi_\eta : B \to \langle\eta\rangle\backslash B$ denotes the canonical projection and $\widetilde{\gamma_0} \in \Upsilon$ is the unique element such that $\widetilde{\gamma_0}^{\#} = \gamma_0$. We will see that φ extends to a local isomorphism

$$\begin{aligned}\Phi : \left(B, \mathcal{O}_B \otimes \mathcal{P}^{\mathbb{C}}\right) &\to \left(\langle\eta\rangle\backslash B\,,\, \widetilde{\mathcal{O}}_B \otimes \mathcal{P}^{\mathbb{C}}\right) \\ &\to \left(\left\langle g^{-1}\gamma_0 g\right\rangle\backslash H\,,\, \widetilde{\mathcal{O}}_H \otimes \mathcal{P}^{\mathbb{C}}\right) \to \left(\langle\gamma_0\rangle\backslash H\,,\, \widetilde{\mathcal{S}}\right) \to \mathcal{X}\end{aligned}$$

of ringed spaces at $0 \mapsto \overline{z_0}$, and therefore Φ^{-1} is a local $\mathcal{P}$-chart of $\mathcal{X}$. The first and the second isomorphism are just induced by $\sqrt[n]{}$ and c .

Lemma 3.1

(i) There exists a unique $\widetilde{z_0} \in_{\mathcal{P}} H$ *such that* $\widetilde{\gamma_0}\widetilde{z_0} = \widetilde{z_0}$ *, and* $\widetilde{z_0}^{\#} = z_0$ *.*

(ii) There exists $\widetilde{g} \in_{\mathcal{P}} H$ *such that* $\widetilde{g}^{\#} = g$ *and* $\widetilde{g}\, i = \widetilde{z_0}$ *.*

Proof: (i) : Let $E := \{g \in G \text{ elliptic }\} \subset G$ and

$$M := \{(g, z) \in E \times H \,|\, gz = z\} .$$

Then M is the preimage of 0 under the smooth map $E \times H \to H$, $(g, z) \mapsto gz - z$ with surjective differential everywhere. One can easily show that M is at the same time the graph of a smooth map $\varphi : E \to H$, this means it is the preimage of 0 under the smooth map $E \times H \to \mathbb{C}$, $(g, z) \mapsto \varphi(g) - z$ with surjective differential everywhere. So $\widetilde{\gamma_0}\widetilde{z_0} = \widetilde{z_0} \Leftrightarrow \widetilde{z_0} = \varphi(\widetilde{\gamma_0})$. $\square$

(ii) : One also can easily compute a smooth map $\varphi : H \to G$ such that $z = \varphi(z)i$ for all $z \in H$. So take $\widetilde{g} := \varphi(\widetilde{z_0})\,\varphi(z_0)^{-1}\, g \in_{\mathcal{P}} G$. $\square$

So $\widetilde{g}^{-1}\widetilde{\gamma_0}\widetilde{g} \in_{\mathcal{P}} K$ since it fixes i , and $\left(\widetilde{g}^{-1}\widetilde{\gamma_0}\widetilde{g}\right)^{2n} = 1$. Therefore by lemma 2.2 since K is commutative we have

$$\widetilde{g}^{-1}\widetilde{\gamma_0}\widetilde{g} = \left(\widetilde{g}^{-1}\widetilde{\gamma_0}\widetilde{g}\right)^{\#} = g^{-1}\gamma_0 g \in K\,.$$

This gives the commuting diagram

$$\begin{array}{ccc} H & \xrightarrow{\widetilde{g}} & H \\ g^{-1}\gamma_0 g \downarrow & \circlearrowleft & \downarrow \widetilde{\gamma_0} \\ H & \xrightarrow[\widetilde{g}]{} & H \end{array}$$

of $\mathcal{P}$-automorphisms inducing the third isomorphism. For the last isomorphism let $U \subset \langle\gamma_0\rangle \backslash H$ be an open neighbourhood of $\pi_{\gamma_0}(z_0)$ such that

$$\overline{\pi}|_U : U \to \overline{\pi}(U) \underset{\text{open}}{\subset} X$$

is biholomorphic. Then for all $V \subset \overline{\pi}(U)$ open $\pi_{\gamma_0}^{-1}\left(\overline{\pi}^{-1}(V) \cap U\right) \subset H$ is already γ_0-invariant, and

$$\pi^{-1}(V) = \dot{\bigcup}_{\gamma \in \Gamma/\langle\gamma_0\rangle} \gamma\, \pi_{\gamma_0}^{-1}\left(\overline{\pi}^{-1}(V) \cap U\right)\,.$$

So similar to the case of a regular point one gets a whole isomorphism $\left(\overline{\pi}|_U\,, \left(|_{\pi_{\gamma_0}^{-1}(\overline{\pi}^{-1}(V)\cap U)}\right)_{V \subset \overline{\pi}(U)\text{ open}}\right)$ of ringed spaces from $\left(U, \widetilde{\mathcal{S}}\Big|_U\right)$ to $\left(\overline{\pi}(U), \mathcal{S}|_{\overline{\pi}(U)}\right)$, where

$$
\begin{aligned}
&|_{\pi_{\gamma_0}^{-1}\left(\overline{\pi}^{-1}(V)\cap U\right)} : \\
&\quad \mathcal{S}(V) = \left\{ f \in \mathcal{O}\left(\pi^{-1}(V)\right) \otimes \mathcal{P}^{\mathbb{C}} \quad \Upsilon\text{-invariant} \right\} \\
&\quad \to \widetilde{\mathcal{S}}\left(\overline{\pi}^{-1}(V) \cap U\right) \\
&= \left\{ f \in \mathcal{O}\left(\pi_{\gamma_0}^{-1}\left(\overline{\pi}^{-1}(V) \cap U\right)\right) \otimes \mathcal{P}^{\mathbb{C}} \,\middle|\, f\left(\widetilde{\gamma_0} z\right) = f \right\}
\end{aligned}
$$

is simply the restriction map, which is an isomorphism of unital $\mathcal{P}^{\mathbb{C}}$-algebras.

Definition 3.2 *$\Phi(0) = \Pi(z_0) \in_{\mathcal{P}} \Upsilon \backslash H$ is called an elliptic point of $\Upsilon \backslash H$. Its body is $\overline{z_0}$, which is an elliptic point of $\Gamma \backslash H$.*

For giving a local trivialization of $\mathcal{F}$ at $\overline{z_0}$ first of all identify the ringed spaces $\left(B, \mathcal{O}_B \otimes \mathcal{P}^{\mathbb{C}}\right)$, $\left(\langle\eta\rangle \backslash B\, ,\, \widetilde{\mathcal{O}}_B \otimes \mathcal{P}^{\mathbb{C}}\right)$, $\left(\left\langle g^{-1}\gamma_0 g\right\rangle \backslash H\, ,\, \widetilde{\mathcal{O}}_H \otimes \mathcal{P}^{\mathbb{C}}\right)$ and $\mathcal{X}$ via Φ locally at $0 \mapsto 0 \mapsto i \mapsto \overline{z_0}$ and define the $\widetilde{\mathcal{O}}_H$- sheaf module $\mathcal{E}_H$ on $\left\langle g^{-1}\gamma_0 g\right\rangle \backslash H$ by

$$
\mathcal{E}_H(V) := \left\{ f \in \mathcal{O}\left(\pi_{g^{-1}\gamma_0 g}^{-1}(V)\right) \,\middle|\, f|_{g^{-1}\gamma_0 g} = f \right\}
$$

for all $V \subset \left\langle g^{-1}\gamma_0 g\right\rangle \backslash H$ open. Now we show that locally at $\overline{z_0}$ we have $\mathcal{S}$- sheaf module isomorphisms

$$
\mathcal{F} \to \mathcal{E}_H \otimes \mathcal{P}^{\mathbb{C}} \to \mathcal{E}_B \otimes \mathcal{P}^{\mathbb{C}} \to \mathcal{O}_B \otimes \mathcal{P}^{\mathbb{C}} ,
$$

where $\mathcal{E}_B$ is a suitable $\widetilde{O}_B$- sheaf module on $\langle\eta\rangle \backslash B$. Similar to Φ the first isomorphism is given by the restriction maps $|_{\pi_{\gamma_0}^{-1}\left(\overline{\pi}^{-1}(V)\cap U\right)}$, $V \subset \overline{\pi}(U)$ open, followed by $|_g$. The second one is given by

$$
|_c : \mathcal{E}_H(V) \to \mathcal{E}_B\left(c^{-1}(V)\right) ,\ f \mapsto f|_c := f\left(c(w)\right) j(c, w)^k
$$

for all $V \subset \left\langle g^{-1}\gamma_0 g\right\rangle \backslash H$, where w denotes the standard holomorphic coordinate on B and $j(c, w) := \frac{1-i}{2} \frac{1}{z-1} \in \mathcal{O}(B)$ is chosen such that $j(c, w)^2 = c'$.

If $2|k$ then $\mathcal{E}_B$ is given by

$$\mathcal{E}_B(V) := \left\{ f \in \mathcal{O}\left(\pi_\eta^{-1}(V)\right) \,\middle|\, f(\eta w)\, \eta^{\frac{k}{2}} = f \right\}$$

and the last isomorphism by

$$\mathcal{E}_B(V) \to \mathcal{O}\left(\left(\sqrt[n]{}\right)^{-1}(V)\right)\,,\ f \mapsto f\left(\sqrt[n]{w}\right) \sqrt[n]{w}^{\,\frac{k}{2}}\, w^{-\left\lceil \frac{k}{2n} \right\rceil}$$

for all $V \subset \langle \eta \rangle \backslash B$ open.

If $2 \not| k$ and $-1 \notin \Gamma$ then automatically $2 \not| n$, and so there exists a unique $\varepsilon \in \langle \eta \rangle$ such that $\varepsilon^2 = \eta$ and so $\operatorname{ord} \varepsilon = n$. Now $\mathcal{E}_B$ is given by

$$\mathcal{E}_B(V) := \left\{ f \in \mathcal{O}\left(\pi_\eta^{-1}(V)\right) \,\middle|\, f(\eta w)\, \varepsilon^k = f \right\}$$

and the last isomorphism by

$$\mathcal{E}_B(V) \to \mathcal{O}\left(\left(\sqrt[n]{}\right)^{-1}(V)\right)\,,\ f \mapsto f\left(\sqrt[n]{w}\right) \sqrt[n]{w}^{\,\frac{k+n}{2}}\, w^{-\left\lceil \frac{k+n}{2n} \right\rceil}$$

for all $V \subset \langle \eta \rangle \backslash B$ open.

At cusps of $\Gamma \backslash H$:

Let $\overline{z_0} \in \Gamma \backslash \partial_{\mathbf{P}^1} H$, $z_0 \in \partial_{\mathbf{P}^1} H$, be a cusp of $\Gamma \backslash H$, and let $g \in G$ such that $g\infty = z_0$ and $\overline{g^{-1}\Gamma g} \cap \overline{P^\infty} = \langle \overline{g_0} \rangle$. Let $\overline{\pi} : \left\langle g g_0 g^{-1} \right\rangle \backslash H \cup \{z_0\} \to X$ denote the canonical projection. Then $\pi = \overline{\pi} \circ \pi_{g g_0 g^{-1}}$, and similar to the case of an elliptic point

$$\begin{aligned} \psi : B &\xrightarrow{\frac{\log}{2\pi i}} \langle g_0 \rangle \backslash H \cup \{\infty\} \xrightarrow{\text{id}} \langle g_0 \rangle \backslash H \cup \{\infty\} \\ &\xrightarrow{g} \left\langle g g_0 g^{-1} \right\rangle \backslash H \cup \{z_0\} \xrightarrow{\overline{\pi}} X \end{aligned}$$

gives a locally biholomorphic map at $0 \mapsto \infty \mapsto \infty \mapsto z_0 \mapsto \overline{z_0}$, so ψ^{-1} is a local chart of X at $\overline{z_0}$.

Now for giving a local $\mathcal{P}$-chart of $\mathcal{X}$ at $\overline{z_0}$ define the sheaf $\widetilde{\mathcal{O}}$ of complex algebras on $\langle \overline{g_0} \rangle \backslash H \cup \{\infty\}$, the sheaf $\widetilde{\mathcal{S}}_\infty$ of $\mathcal{P}^{\mathbb{C}}$-algebras on $\langle \overline{g_0} \rangle \backslash H \cup \{\infty\}$ and the sheaf $\widetilde{\mathcal{S}}_{z_0}$ of $\mathcal{P}^{\mathbb{C}}$-algebras on $\left\langle \overline{g g_0 g^{-1}} \right\rangle \backslash H \cup \{z_0\}$ by

$$\widetilde{\mathcal{O}}(V) := \big\{ f \in \mathcal{O}\left(\pi_{g_0}^{-1}(V \setminus \{\infty\})\right) \,\big|\, f(g_0 z) = f \text{ and}$$
$$f \text{ bounded at } \infty \text{ if } \infty \in V \big\}$$

and

$$\widetilde{\mathcal{S}}_\infty(V) := \Big\{ f \in \mathcal{O}\left(\pi_{g_0}^{-1}(V \setminus \{\infty\})\right) \otimes \mathcal{P}^{\mathbb{C}} \,\Big|\, f(\widetilde{g_0} z) = f \text{ and}$$
$$f \text{ bounded at } \infty \text{ if } \infty \in V \Big\}$$

for all $V \subset \langle \overline{g_0} \rangle \backslash H \cup \{\infty\}$ open, and finally

$$\widetilde{\mathcal{S}}_{z_0}(V) := \Big\{ f \in \mathcal{O}\left(\pi_{g g_0 g^{-1}}^{-1}(V \setminus \{z_0\})\right) \otimes \mathcal{P}^{\mathbb{C}} \,\Big|\, f(\widetilde{g_0} z) = f$$
$$\text{and } f \text{ bounded at } z_0 \text{ if } z_0 \in V \Big\}$$

for all $V \subset \left\langle \overline{g g_0 g^{-1}} \right\rangle \Big\backslash H \cup \{z_0\}$ open, where $\widetilde{g_0} \in g^{-1} \Upsilon g$ is the unique element such that $\widetilde{g_0}^{\#} = g_0$. Again we will see that ψ extends to a local isomorphism

$$\begin{aligned}
\Psi : \left(B, \mathcal{O}_B \otimes \mathcal{P}^{\mathbb{C}}\right) &\to \left(\langle g_0 \rangle \backslash H \cup \{\infty\}, \widetilde{\mathcal{O}} \otimes \mathcal{P}^{\mathbb{C}}\right) \\
&\to \left(\langle g_0 \rangle \backslash H \cup \{\infty\}, \widetilde{\mathcal{S}}_\infty\right) \\
&\to \left(\left\langle g g_0 g^{-1} \right\rangle \backslash H \cup \{z_0\}, \widetilde{\mathcal{S}}_{z_0}\right) \to \mathcal{X}
\end{aligned}$$

of ringed spaces at $0 \mapsto \overline{z_0}$, and therefore Ψ^{-1} is a local $\mathcal{P}$-chart of $\mathcal{X}$ at $\overline{z_0}$. The first isomorphism is induced by $\frac{\log}{2\pi i}$. Now let χ and Ω again be given by lemma 2.5 taken $\widetilde{g_0}$ instead of g . This leads to the commuting diagram

$$\begin{array}{ccc}
H & \xrightarrow{\Omega} & H \\
g_0 \downarrow & \circlearrowleft & \downarrow \widetilde{g_0} \\
H & \xrightarrow[\Omega]{} & H
\end{array}$$

of $\mathcal{P}$-automorphisms inducing the second isomorphism, and the third isomorphism is induced by g as an automorphism of H . The last isomorphism is obtained by the same procedure as the one for elliptic points using an open neighbourhood
$U \subset \left\langle g g_0 g^{-1} \right\rangle \backslash H \cup \{z_0\}$ of z_0 such that

$$\overline{\pi}|_U : U \to \overline{\pi}(U) \underset{\text{open}}{\subset} X$$

is biholomorphic.

Definition 3.3 $\Psi(0) \in_{\mathcal{P}} \mathcal{X}$ *is called a cusp of* $\Upsilon\backslash H$ *. Its body is* $\overline{z_0} \in X$ *, which is a cusp of* $\Gamma\backslash H$ *.*

Observe that in general there is no interpretation of $\Psi(0)$ as a $\mathcal{P}$-point of $\partial_{\mathbf{P}^1} H$.

For giving a local trivialization of $\mathcal{F}$ at $\overline{z_0}$ identify the ringed spaces $\left(B, \mathcal{O}_B \otimes \mathcal{P}^{\mathbb{C}}\right)$, $\left(\langle g_0 \rangle \backslash H \cup \{\infty\}, \widetilde{\mathcal{O}} \otimes \mathcal{P}^{\mathbb{C}}\right)$ and $\mathcal{X}$ via Ψ locally at $0 \mapsto \infty \mapsto \overline{z_0}$. Now we show that locally at $\overline{z_0}$ we have $\mathcal{S}$- sheaf module isomorphisms

$$\mathcal{F} \to \mathcal{E} \otimes \mathcal{P}^{\mathbb{C}} \to \mathcal{O}_B \otimes \mathcal{P}^{\mathbb{C}} ,$$

where $\mathcal{E}$ is a suitable $\widetilde{\mathcal{O}}$- sheaf module on $\langle g_0 \rangle \backslash H$. Similar to Ψ the first isomorphism is given by the restriction maps $|_{\pi_{\gamma_0}^{-1}\left(\overline{\pi}^{-1}(V) \cap U\right)}$, $V \subset \overline{\pi}(U)$ open, followed by $|_g$ and $|_\Omega$.

Definition 3.4 *Assume* $-1 \notin \Gamma$ *. Then either* $g_0 \in g^{-1}\Gamma g$ *or* $-g_0 \in g^{-1}\Gamma g$ *. So the cusp* $\overline{z_0} \in X$ *of* $\Gamma\backslash H$ *is called even (odd) iff* $g_0 \in g^{-1}\Gamma g$ *(resp.* $-g_0 \in g^{-1}\Gamma g$ *).*

If either $2|k$ or $2 \not| k$, $-1 \notin \Gamma$ and $\overline{z_0}$ even then $\mathcal{E}$ is given by

$$\mathcal{E}(V) := \left\{ f \in \mathcal{O}\left(\pi_{g_0}^{-1}(V \setminus \{\infty\})\right) \,\middle|\, f|_{g_0} = f \text{ and } \right.$$
$$\left. f \text{ bounded at } \infty \text{ if } \infty \in V \right\}$$

and since $f|_{g_0}(z) = f(z+1)$ for all $z \in H$ the last isomorphism by

$$\mathcal{E}(V) \to \mathcal{O}\left(\left(\frac{\log}{2\pi i}\right)^{-1}(V)\right) , \; f \mapsto f\left(\frac{\log w}{2\pi i}\right)$$

for all $V \subset \langle g_0 \rangle \backslash H$ open, where w denotes the standard holomorphic coordinate on B .

If $2 \not| k$, $-1 \notin \Gamma$ and $\overline{z_0}$ odd then $\mathcal{E}$ is given by

$$\mathcal{E}(V) := \left\{ f \in \mathcal{O}\left(\pi_{g_0}^{-1}(V \setminus \{\infty\})\right) \,\middle|\, f|_{-g_0} = f \text{ and } \right.$$
$$\left. f \text{ bounded at } \infty \text{ if } \infty \in V \right\}$$

and since $f|_{-g_0}(z) = -f(z+1)$ for all $z \in H$ the last isomorphism by

$$\mathcal{E}(V) \to \mathcal{O}\left(\left(\frac{\log}{2\pi i}\right)^{-1}(V)\right) , \; f \mapsto f\left(\frac{\log w}{2\pi i}\right) e^{-\frac{1}{2}\log w}$$

for all $V \subset \langle g_0 \rangle \backslash H$ open.

4 $\mathcal{P}$-points of the Teichmüller space

Let $g \in \mathbb{N}$ and $\mathcal{T}_g$ be the Teichmüller space for genus g . Let us recall some basic properties. $\mathcal{T}_g$ is a complex domain of dimension

$$N_g = \begin{cases} 0 & \text{if } g = 0 \\ 1 & \text{if } g = 1 \\ 3(g-1) & \text{if } g \geq 2 \end{cases} .$$

For every $\mathbf{a} \in \mathcal{T}_g$ let $S(\mathbf{a})$ be its corresponding compact Riemann surface of genus g . Then all these compact Riemann surfaces $S(\mathbf{a})$, $\mathbf{a} \in \mathcal{T}_g$, glue together to a holomorphic family $\pi : \Xi_g \to \mathcal{T}_g$ over $\mathcal{T}_g$ with $S(\mathbf{a}) := \pi^{-1}(\mathbf{a})$, $\mathbf{a} \in \mathcal{T}_g$, in particular π is a holomorphic submersion, and the moduli space of compact Riemann surfaces of genus g is given by

$$\mathcal{M}_g = \mathcal{T}_g / \Gamma_g$$

with a certain discrete subgroup $\Gamma_g \sqsubset \mathrm{Aut}\mathcal{T}_g$.

Examples 4.1

(i) $\mathcal{T}_0$ consists of one single point with the Riemann sphere $\mathbb{P}^1$ as corresponding compact Riemann surface.

(ii) $\mathcal{T}_1 = H$, $\Gamma_g = SL(2, \mathbb{Z})$, and $S(a) = \mathbb{C}/(\mathbb{Z} + a\mathbb{Z})$ for all $a \in H$.

Now let $\mathbf{a} \in \mathcal{T}_g$ be fixed and $S(\mathbf{a})$ be given by $U_1, \ldots, U_n \subset \mathbb{C}$ open together with the glueing data

$$\sigma_{ij} : U_i \underset{\text{open}}{\supset} U_{ij} \to U_{ji} \underset{\text{open}}{\subset} U_j$$

biholomorphic. Then of course $(U_i)_{i=1,\ldots,n}$ is an open cover of $S(\mathbf{a})$, and after some refinement of this open cover we may assume that $H^1\left((U_i)_{i\in\{1,\ldots,n\}}, TX\right) \simeq H^1(X, TX) \simeq T_{\mathbf{a}}\mathcal{T}_g$ canonically, and then using charts of Ξ_g where the projection π is just given by projecting onto the first N_g coordinates we see that there exist an open neighbourhood B of $\mathbf{a}$ in $\mathcal{T}_g$ and families $\left(U_{ij}^{(\mathbf{w})}\right)_{\mathbf{w}\in B}$, $\left(\sigma_{ij}^{(\mathbf{w})}\right)_{\mathbf{w}\in B}$, $i, j = 1, \ldots, n$, such that

{i}
$$U_{ij}^{(\mathbf{w})} \subset U_i$$

is open and

$$\sigma_{ij}^{(\mathbf{w})} : U_{ij}^{(\mathbf{w})} \to U_{ji}^{(\mathbf{w})}$$

is biholomorphic for all $i, j = 1, \ldots, n$ and $\mathbf{w} \in B$,

{ii}
$$\dot{\bigcup}_{\mathbf{w}\in B} U_{ij}^{(\mathbf{w})} \subset U_i \times B$$

is open and

$$\dot{\bigcup}_{\mathbf{w}\in B} U_{ij}^{(\mathbf{w})} \to \mathbb{C}\,,\ (z, \mathbf{w}) \mapsto \sigma_{ij}^{(\mathbf{w})}(z)$$

is holomorphic for all $i, j = 1, \ldots, n$,

{iii} $U_{ij}^{(\mathbf{a})} = U_{ij}$ and $\sigma_{ij}^{(\mathbf{a})} = \sigma_{ij}$ for all $i, j = 1, \ldots, n$, and finally

{iv} $S(\mathbf{w})$ is given by the glueing data

$$\sigma_{ij}^{(\mathbf{w})} : U_i \underset{\text{open}}{\supset} U_{ij}^{(\mathbf{w})} \to U_{ji}^{(\mathbf{w})} \underset{\text{open}}{\subset} U_j$$

for all $\mathbf{w} \in B$, and so in particular we have a $\mathbb{C}$-linear map

$$\begin{aligned} \Omega : T_{\mathbf{a}}\mathcal{T}_g &\to Z^1\left((U_i)_{i\in\{1,\ldots,n\}}, TX\right) , \\ \mathbf{v} &\mapsto \left(\left(\sigma'_{ij}\right)^{-1}\left(\partial_t \sigma_{ij}^{(\mathbf{a}+t\mathbf{v})}\Big|_{t=0}\right)\right)_{i,j\in\{1,\ldots,n\}} \end{aligned}$$

such that $[\ ]\circ\Omega : T_{\mathbf{a}}\mathcal{T}_g \to H^1(X, TX)$ is an isomorphism, where

$$[\ ] : Z^1\left((U_i)_{i\in\{1,\ldots,n\}}, TX\right) \to H^1\left((U_i)_{i\in\{1,\ldots,n\}}, TX\right) \hookrightarrow H^1(X, TX)$$

denotes the canonical projection.

Now let $\widetilde{\mathbf{a}} \in_{\mathcal{P}} B$ such that $\widetilde{\mathbf{a}}^{\#} = \mathbf{a}$. Then according to (vii) of section 1 we can assign to $\widetilde{\mathbf{a}}$ the $\mathcal{P}$- RIEMANN surface $S\left(\widetilde{\mathbf{a}}\right) = \pi^{-1}\left(\widetilde{\mathbf{a}}\right)$. $S\left(\widetilde{a}\right)$ is given by the local $\mathcal{P}$-charts $U_1, \ldots, U_n \subset \mathbb{C}$ with the $\mathcal{P}$- glueing data

$$\sigma_{ij}^{\widetilde{a}} : U_i \underset{\text{open}}{\supset} U_{ij} \to_{\mathcal{P}} U_{ji} \underset{\text{open}}{\subset} U_j \,,$$

which are $\mathcal{P}$-isomorphisms, and clearly its body is $S(a)$. The purpose of this section is to show that any $\mathcal{P}$- RIEMANN surface $\mathcal{X}$ with compact body $X := \mathcal{X}^{\#}$ of genus g can be realized as a $\mathcal{P}$-point of the TEICHMÜLLER space $\mathcal{T}_g$, which is of general interest. Before doing so we need a lemma:

Lemma 4.2

(i) Let $\mathcal{X}$ be a $\mathcal{P}$- RIEMANN *surface such that $X := \mathcal{X}^{\#}$ is compact of genus $g \geq 2$, and let Φ be a $\mathcal{P}$-automorphism of $\mathcal{X}$ with $\Phi^{\#} = \mathrm{id}$. Then $\Phi = \mathrm{id}$.*

(ii) Let $a \in_{\mathcal{P}} H = \mathcal{T}_1$. Then the $\mathcal{P}$-automorphisms of $S(a) = \mathbb{C}/(\mathbb{Z} + a\mathbb{Z})$ with id *as body are given by the translations*

$$t_b : \mathbb{C}/(\mathbb{Z} + a\mathbb{Z}) \to_{\mathcal{P}} \mathbb{C}/(\mathbb{Z} + a\mathbb{Z}) \,,\, z \mapsto z + b \,,$$

$b \in \mathcal{I}^{\mathbb{C}}$, where $\mathcal{I} \lhd \mathcal{P}$ denotes the unique maximal ideal in $\mathcal{P}$.

Recall that the $\mathcal{P}$- RIEMANN surface $S(a) = \mathbb{C}/(\mathbb{Z}+a\mathbb{Z})$ in (ii) can be written as $\left(\mathbb{C}\left/\left(\mathbb{Z} + a^{\#}\mathbb{Z}\right)\right. , \mathcal{S}\right)$, where $\mathcal{S}$ is given by

$$\mathcal{S}(U) := \left\{ f \in \mathcal{O}\left(\pi^{-1}(U)\right) \otimes \mathcal{P}^{\mathbb{C}} \,\middle|\, f(z + m + na) = f(z) \text{ for all } m, n \in \mathbb{Z} \right\}$$

for all $U \subset \mathbb{C}\left/\left(\mathbb{Z} + a^{\#}\mathbb{Z}\right)\right.$ open and $\pi : \mathbb{C} \to \mathbb{C}\left/\left(\mathbb{Z} + a^{\#}\mathbb{Z}\right)\right.$ denotes the canonical projection.

Proof: via induction on N , where $\mathcal{I}^N = 0$, $\mathcal{I}$ being the unique maximal ideal of $\mathcal{P}$. If $N = 1$ then $\mathcal{P} = \mathbb{R}$, and so both assertions are trivial. Now assume $\mathcal{I}^{N+1} = 0$, and define

$$\mathcal{Q} := \mathcal{P}\left/\mathcal{I}^N\right.$$

with unique maximal ideal $\mathcal{J} := \mathcal{I}\left/\mathcal{I}^N\right. \lhd \mathcal{Q}$ having $\mathcal{J}^N = 0$, and let $\natural : \mathcal{P} \to \mathcal{Q}$ be the canonical projection.

For proving (i) let $\mathcal{X}$ be a $\mathcal{P}$- RIEMANN surface with body $X := \mathcal{X}^{\#}$, compact of genus ≥ 2 , and Φ be a $\mathcal{P}$-automorphism of $\mathcal{X}$ with $\Phi^{\#} = \mathrm{id}$. Let $\mathcal{X}$ be given by $U_1, \ldots, U_n \subset \mathbb{C}$ open together with the $\mathcal{P}$- glueing data

$$\varphi_{ij} : U_i \underset{\text{open}}{\supset} U_{ij} \to_{\mathcal{P}} U_{ji} \underset{\text{open}}{\subset} U_j \,.$$

Then in the local $\mathcal{P}$-charts U_i , $i = 1, \ldots, n$, Φ is given by $\mathcal{P}$-automorphisms $\Phi_i : U_i \to_{\mathcal{P}} U_i$ having $\Phi_i^{\#} = \mathrm{id}$ and

$$\begin{array}{ccccc} & U_{ij} & \xrightarrow{\varphi_{ij}} & U_{ji} & \\ \Phi_i|_{U_{ij}} \uparrow & & \circlearrowleft & & \uparrow \Phi_j|_{U_{ji}} \\ & U_{ij} & \xrightarrow[\varphi_{ij}]{} & U_{ji} & \end{array} .$$

By induction hypothesis $\Phi^\natural = \mathrm{id}$, so $\Phi_i = \mathrm{id} + f_i$ with suitable $f_i \in \mathcal{O}(U_{ij}) \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^N$, and a straight forward calculation using $\mathcal{I}^{N+1} = 0$ shows that f_i glue together to an element $f \in H^0(X, TX) \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^N$, but $H^0(X, TX) = 0$ since X is of genus ≥ 2 . So all $\Phi_i = \mathrm{id}$, and so $\Phi = \mathrm{id}$.

For proving (ii) let $a \in_{\mathcal{P}} H$ and Φ be a $\mathcal{P}$-automorphism of $S(a) = \mathbb{C}/(\mathbb{Z} + a\mathbb{Z})$. By induction hypothesis we already know that there exists $c \in \mathcal{I}^{\mathbb{C}}$ such that $\Phi^\natural = t_{c^\natural}$. Then the lift of Φ as a $\mathcal{P}$-automorphism of $\mathbb{C}$ commuting with translations by $\mathbb{Z} + a\mathbb{Z}$ is given by $z \mapsto z + c + f$ with a suitable $f \in \mathcal{O}(\mathbb{C}) \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^N$. But then f has to be invariant under translations by $\mathbb{Z} + a^{\#}\mathbb{Z}$, and so it has to be constant $\in \left(\mathcal{I}^{\mathbb{C}}\right)^N$. Define $b := c + f$. $\Box$

Theorem 4.3 *Let $\mathcal{X}$ be a $\mathcal{P}$- Riemann surface, whose body $X = S(\mathbf{a})$ is a compact* Riemann *surface of genus g , $\mathbf{a} \in \mathcal{T}_g$. Then there exists a unique $\widetilde{\mathbf{a}} \in_{\mathcal{P}} \mathcal{T}_g$ such that*

(i) $\widetilde{\mathbf{a}}^{\#} = \mathbf{a}$ *and*

(ii) there exists a $\mathcal{P}$-isomorphism

$$\Phi : S(\widetilde{\mathbf{a}}) \to_{\mathcal{P}} \mathcal{X}$$

having $\Phi^{\#} = \mathrm{id}$.

By lemma 4.2 Φ is uniquely determined by $\mathcal{X}$ if and only if $g \geq 2$.

Proof: again via induction on N , where $\mathcal{I}^N = 0$, $\mathcal{I}$ being the unique maximal ideal of $\mathcal{P}$. If $N = 1$ then $\mathcal{P} = \mathbb{R}$, and so the assertion is again trivial. So assume $\mathcal{I}^{N+1} = 0$, and let $\mathcal{X}$ be a $\mathcal{P}$- Riemann surface with body $X := \mathcal{X}^{\#} = S(\mathbf{a})$, $\mathbf{a} \in \mathcal{T}_g$. Let $\mathcal{X}$ be given by $U_1, \ldots, U_n \subset \mathbb{C}$ open together with the $\mathcal{P}$- glueing data

$$\varphi_{ij} : U_i \underset{\text{open}}{\supset} U_{ij} \to_{\mathcal{P}} U_{ji} \underset{\text{open}}{\subset} U_j .$$

Then $(U_i)_{i\in\{1,\dots,n\}}$ forms an open cover of X , and after maybe some refinement of this open cover we may again assume that

$$H^1\left((U_i)_{i=1,\dots,n}, TX\right) \simeq H^1(X,TX) \simeq T_{\mathbf{a}}\mathcal{T}_g$$

canonically. So let B be an open neighbourhood of $\mathbf{a}$ in $\mathcal{T}_g$ and $\left(U_{ij}^{(\mathbf{w})}\right)_{\mathbf{w}\in B}$ and $\left(\sigma_{ij}^{(\mathbf{w})}\right)_{\mathbf{w}\in B}$ be families such that the conditions {i} - {iv} are fulfilled. Again define $\mathcal{Q} := \mathcal{P}/\mathcal{I}^N$ with unique maximal ideal $\mathcal{J} := \mathcal{I}/\mathcal{I}^N \lhd \mathcal{Q}$ having $\mathcal{J}^N = 0$, and let $\natural : \mathcal{P} \to \mathcal{Q}$ be the canonical projection. Let $X^\natural$ be the $\mathcal{Q}$- Riemann surface given by the $\mathcal{Q}$- glueing data

$$\varphi_{ij}^\natural : U_i \underset{\text{open}}{\supset} U_{ij} \to_{\mathcal{P}} U_{ji} \underset{\text{open}}{\subset} U_j\,.$$

Then evidently $\left(\mathcal{X}^\natural\right)^\# = X$, and therefore by induction hypothesis there exists a unique $\mathbf{b} \in_{\mathcal{P}} B$ with $\mathbf{b}^\# = \mathbf{a}$ such that there exists a $\mathcal{Q}$-isomorphism $\Psi : S\left(\mathbf{b}^\natural\right) \to_{\mathcal{Q}} \mathcal{X}^\natural$ having $\Psi^\# = \mathrm{id}$. In the local $\mathcal{Q}$-charts U_i , $i = 1, \dots, n$, Ψ is given by $\mathcal{Q}$-automorphisms $\Psi_i : U_i \to_{\mathcal{Q}} U_i$ having $\Psi^\# = \mathrm{id}$ and

$$\begin{array}{ccc} U_{ij} & \xrightarrow{\varphi_{ij}^\natural} & U_{ji} \\ \Psi_i|_{U_{ij}} \uparrow & \circlearrowleft & \uparrow \Psi_j|_{U_{ji}} \\ U_{ij} & \xrightarrow[\sigma_{ij}^{\left(\mathbf{b}^\natural\right)}]{} & U_{ji} \end{array}\,.$$

Let $\widetilde{\Psi_i} : U_i \to_{\mathcal{P}} U_i$, $i = 1, \dots, n$, be arbitrary such that $\widetilde{\Psi_i}^\natural = \Psi_i$, and so $\widetilde{\Psi_i}$, $i = 1, \dots, n$, are automatically $\mathcal{P}$-automorphisms, and define

$$\rho_{ij} := \left(\left.\widetilde{\Psi_j}\right|_{U_{ji}}\right)^{-1} \circ \varphi_{ij} \circ \left.\widetilde{\Psi_i}\right|_{U_{ij}} : U_{ij} \to_{\mathcal{P}} U_{ji}\,.$$

Then for all $i, j = 1, \dots, n$ since $\rho_{ij}^\natural = \sigma_{ij}^{\left(\mathbf{b}^\natural\right)} = \left(\sigma_{ij}^{(\mathbf{b})}\right)^\natural$ and $\sigma_{ij}^{(\mathbf{b})}$ is a $\mathcal{P}$-automorphism we see that

$$\rho_{ij}(z) = \sigma_{ij}^{(\mathbf{b})}\left(z + \omega_{ij}(z)\right)$$

with some suitable $\omega_{ij} \in \mathcal{O}\left(U_{ij}\right) \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^N$, $i, j \in \{1, \dots, n\}$. Now an easy calculation shows that the ω_{ij} , $i, j \in \{1, \dots, n\}$, form an element $\omega \in Z^1\left((U_i)_{i\in\{1,\dots,n\}}, TX\right) \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^N$, and therefore there exist $\mathbf{V} \in T_{\mathbf{a}}\mathcal{T}_g \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^N$ and $\mu \in C^0\left((U_i)_{i\in\{1,\dots,n\}}, TX\right) \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^N$ such that $\omega = \Omega(\mathbf{V}) + \delta\mu$, where δ denotes the coboundary operator on the cocomplex associated to the sheaf cohomology of TX and the open cover $X = \bigcup_{i=1,\dots,n} U_i$. In the local charts U_i , $i = 1, \dots, n$, μ is given by some $\mu_i \in \mathcal{O}\left(U_i\right) \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^N$, $i = 1, \dots, n$. Now define $\widetilde{\mathbf{a}} := \mathbf{b} + \mathbf{V}$ and

$$\Phi_i : U_i \to_{\mathcal{P}} U_i \, , \, z \mapsto \widetilde{\Psi_i}(z) - \mu_i(z) \, ,$$

$i = 1, \dots, n$. Then $\widetilde{\mathbf{a}}^{\#} = \mathbf{a}$, $\Phi_i^{\#} = \mathrm{id}$, and so Φ_i , $i = 1, \dots, n$, are automatically $\mathcal{P}$-automorphisms. A straight forward calculation shows that

$$\begin{array}{ccccc} & U_{ij} & \xrightarrow{\varphi_{ij}} & U_{ji} & \\ \Phi_i|_{U_{ij}} \uparrow & & \circlearrowleft & & \uparrow \Phi_j|_{U_{ji}} \\ & U_{ij} & \xrightarrow[\sigma_{ij}^{(\widetilde{\mathbf{a}})}]{} & U_{ji} & \end{array} \, ,$$

and so Φ_i , $i = 1, \dots, n$, glue together to a $\mathcal{P}$-isomorphism $\Phi : S(\widetilde{\mathbf{a}}) \to_{\mathcal{P}} \mathcal{X}$ with $\Phi^{\#} = \mathrm{id}$. This proves the existence of $\widetilde{\mathbf{a}}$.

For proving uniqueness we may assume without loss of generality that $g \geq 1$. Let also $\mathbf{c} \in_{\mathcal{P}} B$ such that $\mathbf{c}^{\#} = \mathbf{a}$, and let $\Lambda : S(\widetilde{\mathbf{a}}) \to_{\mathcal{P}} S(\mathbf{c})$ be a $\mathcal{P}$-isomorphism such that $\Lambda^{\#} = \mathrm{id}$. We will show that $\mathbf{c} = \widetilde{\mathbf{a}}$. By induction hypothesis we already know that $\widetilde{\mathbf{a}}^{\natural} = \mathbf{c}^{\natural}$, so $\mathbf{W} := \mathbf{c} - \widetilde{\mathbf{a}} \in T_{\mathbf{a}}\mathcal{T}_g \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^N$, and $\Lambda^{\natural}$ is a $\mathcal{Q}$-automorphism of $S(\widetilde{\mathbf{a}})^{\natural} = S\left(\widetilde{\mathbf{a}}^{\natural}\right)$.

First case: $g \geq 2$.

Then by lemma 4.2 (ii) we directly know that $\Lambda^{\natural} = \mathrm{id}$, and so in the local charts U_i , $i = 1, \dots, n$, Λ is given by $\mathcal{P}$-automorphisms Λ_i of U_i with

$$\begin{array}{ccccc} & U_{ij} & \xrightarrow{\sigma_{ij}^{(\mathbf{c})}} & U_{ji} & \\ \Lambda_i|_{U_{ij}} \uparrow & & \circlearrowleft & & \uparrow \Lambda_j|_{U_{ji}} \\ & U_{ij} & \xrightarrow[\sigma_{ij}^{(\widetilde{\mathbf{a}})}]{} & U_{ji} & \end{array} \, ,$$

and $\Lambda_i^{\natural} = \mathrm{id}$. Therefore $\Lambda_i = \mathrm{id} + f_i$ with suitable $f_i \in \mathcal{O}(U_i) \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^N$.

Second case: $g = 1$.

By lemma 4.2 (i) we know that $\Lambda^{\natural} = t_{d^{\natural}}$ with some $d \in \mathcal{I}^{\mathbb{C}}$. So

$$\Lambda \circ t_{-d} : S(\widetilde{\mathbf{a}}) \to_{\mathcal{P}} S(\mathbf{c})$$

is a $\mathcal{P}$-isomorphism having $(\Lambda \circ t_{-d})^{\natural} = \mathrm{id}$. So again in the local charts U_i , $i = 1, \dots, n$, $\Lambda \circ t_{-d}$ is given by $\mathrm{id} + f_i$ where $f_i \in \mathcal{O}(U_i) \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^N$.

In both cases now an easy calculation shows that $\Omega(\mathbf{W}) = \delta f$, $f := (f_i)_{i=1,\ldots,n}$ considered as an element of $C^0\left((U_i)_{i=1,\ldots,n}, TX\right) \otimes \left(\mathcal{I}^{\mathbb{C}}\right)^N$. Therefore $\overline{\Omega}(\mathbf{W}) = \mathbf{0}$, and so $\mathbf{W} = \mathbf{0}$. $\square$

Corollary 4.4 *Let $\mathcal{X}$ be a $\mathcal{P}$- RIEMANN surface with body $\mathcal{X}^\# = \mathbb{P}^1$. Then there exists a $\mathcal{P}$-isomorphism $\Phi : \mathbb{P}^1 \to_{\mathcal{P}} \mathcal{X}$ having $\Phi^\# = \mathrm{id}$.*

5 The main result

Now we return to the backbone of the article. So let $\mathcal{X} = (X, \mathcal{S})$, $X := \Gamma\backslash H \cup \{$ cusps of $\Gamma\backslash H\}$, be the $\mathcal{P}$- RIEMANN surface constructed in section 3 , and let g be the genus of X . Then by theorem 4.3 we may identify $\mathcal{X}$ with $S(\widetilde{\mathbf{a}})$ and so X with $S(\mathbf{a})$ for some $\widetilde{\mathbf{a}} \in_{\mathcal{P}} \mathcal{T}_g$ and $\mathbf{a} := \widetilde{\mathbf{a}}^\#$.

Let $\widetilde{e_1}, \ldots, \widetilde{e_R} \in_{\mathcal{P}} \mathcal{X} = S(\widetilde{\mathbf{a}})$, $\rho = 1, \ldots, R$, be the elliptic points and $\widetilde{s_1}, \ldots, \widetilde{s_S} \in_{\mathcal{P}} \mathcal{X}$, $\sigma = 1, \ldots, S$, the cusps of $\Upsilon\backslash H$. Then $e_\rho := \widetilde{e_\rho}^\# \in X = S(\mathbf{a})$, $\rho = 1, \ldots, R$, $s_\sigma := \widetilde{s_\sigma}^\# \in X$, $\sigma = 1, \ldots, S$, are automatically the elliptic points resp. cusps of $\Gamma\backslash H$. Let $n_\rho \in \mathbb{N}$ denote the period of the elliptic point e_ρ , $\rho = 1, \ldots, R$. Let $U_\rho \subset X$ and $V_\sigma \subset X$ be pairwise disjoint open connected coordinate neighbourhoods of e_ρ , $\rho = 1, \ldots, R$, and s_σ , $\sigma = 1, \ldots, S$, resp.. Then via common local charts on $S(\mathbf{w})$, $\mathbf{w} \in W$, $W \subset \mathcal{T}_g$ a suitably small open neighbourhood of $\mathbf{a}$, we can identify the sets $W \times U_\rho$, $\rho = 1, \ldots, R$, $W \times V_\sigma$, $\sigma = 1, \ldots, S$, with pairwise disjoint open sets of Ξ_g such that

$$\begin{array}{ccc} W \times U_\rho & \hookrightarrow & \Xi_g \\ \mathrm{pr}_1 \downarrow & \circlearrowleft & \downarrow \pi \\ W & \subset & \mathcal{T}_g \end{array} ,$$

$\rho = 1, \ldots, R$, and

$$\begin{array}{ccc} W \times V_\sigma & \hookrightarrow & \Xi_g \\ \mathrm{pr}_1 \downarrow & \circlearrowleft & \downarrow \pi \\ W & \subset & \mathcal{T}_g \end{array} ,$$

$\sigma = 1, \ldots, S$. This gives us at the same time embeddings $U_\rho, V_\sigma \hookrightarrow S(\mathbf{w})$, $\rho = 1, \ldots, R$, $\sigma = 1, \ldots S$, $\mathbf{w} \in W$, as pairwise disjoint open sets. Now let

$$\widetilde{E} := (\widetilde{e_1}, \ldots, \widetilde{e_R}, \widetilde{s_1}, \ldots, \widetilde{s_S}) \in_{\mathcal{P}} \mathcal{X}^{R+S}$$

and

$$\mathcal{U} := U_1 \times \cdots \times U_R \times V_1 \times \cdots \times V_S \subset X^{R+S} \,,$$

which is an open neighbourhood of
$E := \widetilde{E}^{\#} = (e_1, \ldots, e_R, s_1, \ldots, s_S) \in X^{R+S}$, and at the same time for all $\mathbf{w} \in W$ it is identified with some open set in $S(\mathbf{w})^{R+S}$. Let

$$\Xi_g|_W := \pi^{-1}(W) = \dot{\bigcup}_{\mathbf{w}\in W} S(\mathbf{w}) \,.$$

Then $(\pi, \mathrm{id}) : \Xi_g|_W \times \mathcal{U} \to W \times \mathcal{U}$ is a familiy of compact RIEMANN surfaces $S(\mathbf{w}) \times \{\mathbf{u}\} = S(\mathbf{w})$, $(\mathbf{w}, \mathbf{u}) \in W \times \mathcal{U}$. Furthermore let

$$U_0 := \dot{\bigcup}_{(\mathbf{w},\mathbf{u})\in W\times\mathcal{U}} \left(S(\mathbf{w}) \setminus \{u_1, \ldots, u_R, v_1, \ldots, v_S\}\right) \times \{\mathbf{u}\} \subset \Xi_g|_W \times \mathcal{U}$$

open and dense, where $\mathbf{u} = (u_1, \ldots, u_R, v_1, \ldots, v_S)$. Then

$$\Xi_g|_W \times \mathcal{U} = U_0 \cup \bigcup_{\rho=1}^{R} (W \times U_\rho \times \mathcal{U}) \cup \bigcup_{\sigma=1}^{S} (W \times V_\sigma \times \mathcal{U})$$

is a finite open cover, and we can define holomorphic line bundles on $\Xi_g|_W \times \mathcal{U}$ via trivializations on U_0 and on each $W \times U_\rho \times \mathcal{U}$, $\rho = 1, \ldots, R$, and $W \times V_\sigma \times \mathcal{U}$, $\sigma = 1, \ldots, S$, and transition functions $\phi_{0\rho} \in \mathcal{O}\left(U_0 \cap (W \times U_\rho \times \mathcal{U})\right)$, $\rho = 1, \ldots, R$, resp. $\psi_{0\sigma} \in \mathcal{O}\left(U_0 \cap (W \times V_\rho \times \mathcal{U})\right)$, $\sigma = 1, \ldots, S$.

Furthermore let T^*_{rel} denote the relative cotangent bundle of the family $\Xi_g|_W \times \mathcal{U} \to W \times \mathcal{U}$ of compact RIEMANN surfaces, see for example section 10.1 of [9] . It is a holomorphic line bundle on $\Xi_g|_W \times \mathcal{U}$ such that $T^*_{\mathrm{rel}}|_{S(\mathbf{w})\times\{\mathbf{u}\}}$ is the cotangent bundle of $S(\mathbf{w})$ for all $(\mathbf{w}, \mathbf{u}) \in W \times \mathcal{U}$. Therefore $T^*_{\mathrm{rel}}|_{S(\mathbf{w})\times\{\mathbf{u}\}} = T^*S(\mathbf{w})$ even for all $\mathcal{P}$-points $(\mathbf{w}, \mathbf{u}) \in_{\mathcal{P}} W \times \mathcal{U}$.

Finally let $\mathcal{F}$ denote the holomorphic $\mathcal{P}$- line bundle over $\mathcal{X} = S\left(\widetilde{\mathbf{a}}\right)$ defined in section 3 having $M_k(\Upsilon) = H^0(\mathcal{F})$.

First we treat the case $2|k$.

We define the holomorphic line bundle $L_k \to \Xi_g|_W \times \mathcal{U}$ by the transition functions $\phi_{0\rho}(z) := (z - u_\rho)^{-\frac{k}{2} + \left\lceil \frac{k}{2n_\rho} \right\rceil}$, $\rho = 1, \ldots, R$, and $\psi_{0\sigma}(z) := (z - v_\sigma)^{-\frac{k}{2}}$, $\sigma = 1, \ldots, S$, where z denotes a local coordinate on U_ρ resp. V_σ . Then for each $(\mathbf{w}, \mathbf{u}) \in W \times \mathcal{U}$ the holomorphic sections of $L_k|_{S(\mathbf{w})\times\{\mathbf{u}\}}$ are precisely the meromorphic functions on $S(\mathbf{w})$ with poles at the points $u_\rho \in U_\rho \hookrightarrow S(\mathbf{w})$ of order at most $\frac{k}{2} - \left\lceil \frac{k}{2n_\rho} \right\rceil$, $\rho = 1, \ldots, R$, and

poles at the points $v_\sigma \in V_\sigma \hookrightarrow S(\mathbf{w})$, $\sigma = 1, \ldots, S$, of order at most $\frac{k}{2}$ and holomorphic at all other points of $S(\mathbf{w})$.

Furthermore we define the holomorphic line bundle $C \to \Xi_g|_W \times \mathcal{U}$ by the transition functions $\phi_{0\rho}(z) := 1$, $\rho = 1, \ldots, R$, and $\psi_{0\sigma}(z) := z - v_\sigma$, $\sigma = 1, \ldots, S$. Then for each $(\mathbf{w}, \mathbf{u}) \in W \times \mathcal{U}$ the holomorphic sections of $C|_{S(\mathbf{w}) \times \{\mathbf{u}\}}$ are precisely the holomorphic functions on $S(\mathbf{w})$ vanishing at $v_\sigma \in V_\sigma \hookrightarrow S(\mathbf{w})$, $\sigma = 1, \ldots, S$. Therefore clearly $\deg C|_{S(\mathbf{w}) \times \{\mathbf{u}\}} = -S$ for all $(\mathbf{w}, \mathbf{u}) \in W \times \mathcal{U}$, and we identify the sections of C with ordinary holomorphic functions on $\Xi_g|_W \times \mathcal{U}$ vanishing on $\{z = v_\sigma\}$, $\sigma = 1, \ldots, S$.

Finally we define the line bundle $L'_k := {T^*_{\text{rel}}}^{\otimes \frac{k}{2}} \otimes L_k$ over $\Xi_g|_W \times \mathcal{U}$.

Lemma 5.1 *We have isomorphisms* $\mathcal{F} \simeq L'_k|_{S(\widetilde{\mathbf{a}}) \times \{\widetilde{E}\}}$ *and so* $M_k(\Upsilon) \simeq H^0\left(L'_k|_{S(\widetilde{\mathbf{a}}) \times \{\widetilde{E}\}}\right)$ *, the last isomorphism mapping* $S_k(\Upsilon)$ *precisely to* $H^0\left((L'_k \otimes C)|_{S(\widetilde{\mathbf{a}}) \times \{\widetilde{E}\}}\right)$.

Proof: Since $g' = j(g, z)^2$ for all $g \in_{\mathcal{P}} G$ regarded as a $\mathcal{P}$-automorphism of H , identifying the trivial and the cotangent bundle on H we see that $\mathcal{F} \simeq (T^*\mathcal{X})^{\otimes \frac{k}{2}} = {T^*_{\text{rel}}}^{\otimes \frac{k}{2}}\Big|_{S(\widetilde{\mathbf{a}}) \times \{\widetilde{E}\}}$ on $X' := (\Gamma \backslash H) \setminus \{e_1, \ldots, e_r\}$.

Now let $\Phi_1, \ldots, \Phi_R$ and $\Psi_1, \ldots, \Psi_S$ denote the local $\mathcal{P}$-charts of $\mathcal{X}$ at $e_1, \ldots, e_R$ and $s_1, \ldots, s_S$ resp. given in section 3 . Then via these local $\mathcal{P}$-charts the elliptic points $\widetilde{e}_1, \ldots, \widetilde{e}_R$ and the cusps $\widetilde{s}_1, \ldots, \widetilde{s}_S \in_{\mathcal{P}} \mathcal{X}$ of $\Upsilon \backslash H$ are identified with the ordinary point $0 \in B$.

A straight forward calculation shows that the holomorphic sections of $(T^*\mathcal{X})^{\otimes \frac{k}{2}}$ regarded as holomorphic sections of $\mathcal{F}$ on X' vanish in the local $\mathcal{P}$-chart Φ_ρ at 0 of order $\frac{k}{2} - \left\lceil \frac{k}{2n} \right\rceil$ for all $\rho = 1, \ldots, R$ and in the local $\mathcal{P}$-chart Ψ_σ at 0 of order $\frac{k}{2}$ for all $\sigma = 1, \ldots, S$. This proves the first statement.

Furthermore if $f \in M_k(\Upsilon) = H^0(\mathcal{F})$ one sees that $f \in S_k(\Upsilon)$ iff f vanishes in the local $\mathcal{P}$-chart Ψ_σ at 0 for all $\sigma = 1, \ldots, S$, which proves the second statement. $\square$

Observe that

$$\begin{aligned} \deg L'_k|_{S(\mathbf{w}) \times \{\mathbf{u}\}} &= \frac{k}{2} \deg T^*_{\text{rel}}|_{S(\mathbf{w}) \times \{\mathbf{u}\}} + \deg L_k|_{S(\mathbf{w}) \times \{\mathbf{u}\}} \\ &= k(g-1) + \frac{k}{2}(R+S) - \sum_{\rho=1}^{R} \left\lceil \frac{k}{2n_\rho} \right\rceil \qquad (1) \end{aligned}$$

is independent of the point $(\mathbf{w},\mathbf{u}) \in W \times \mathcal{U}$.

Lemma 5.2

$$\dot{\bigcup}_{(\mathbf{w},\mathbf{u})\in W\times\mathcal{U}} H^0\left(L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}}\right)$$

is a holomorphic vector bundle over $W \times \mathcal{U}$ *, containing*

$$\dot{\bigcup}_{(\mathbf{w},\mathbf{u})\in W\times\mathcal{U}} H^0\left((L'_k \otimes C)|_{S(\mathbf{w})\times\{\mathbf{u}\}}\right)$$

as a holomorphic sub vector bundle.

Proof: We will use theorem 5 in section 10.5 of [4] , which says the following:

> If $\dim_{\mathbb{C}} H^i\left(X_y, \underline{V}_y\right)$ is independent of $y \in Y$ then all sheaves $f_{(i)}\left(\underline{V}\right)$ are locally free and all maps
>
> $$f_{y,i} : f_{(i)}\left(\underline{V}\right) / \mathfrak{m}_y f_{(i)}\left(\underline{V}\right) \to H^i\left(X_y, \underline{V}_y\right)$$
>
> are isomorphisms.

Hereby $f : X \to Y$ denotes a holomorphic family of compact complex manifolds $X_y := f^{-1}(y)$, $y \in Y$, $\underline{V}$ a holomorphic vector bundle over X and $f_{(i)}\left(\underline{V}\right)$, $i \in \mathbb{N}$, the higher direct image sheaves of $\underline{V}$ under f . $\underline{V}_y := \underline{V}|_{X_y}$, $\mathfrak{m}_y \lhd \mathcal{O}_Y$ denotes the maximal ideal of holomorphic functions on Y vanishing at the point $y \in Y$, and finally $f_{y,i} : f_{(i)}\left(\underline{V}\right) / \mathfrak{m}_y f_{(i)}\left(\underline{V}\right) \to H^i\left(X_y, \underline{V}_y\right)$ denotes the canonical homomorphism. Recall that the sheaf $f_{(0)}\left(\underline{V}\right)$ is given by the assignment $U \mapsto H^0\left(f^{-1}(U), \underline{V}\right)$ for all $U \subset Y$ open.

So we have to show that $\dim H^0\left(L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}}\right)$ and $\dim H^0\left((L'_k \otimes C)|_{S(\mathbf{w})\times\{\mathbf{u}\}}\right)$ are independent of the point $(\mathbf{w},\mathbf{u}) \in W \times \mathcal{U}$. We use formula (1) . The case $g = 0$ is trivial.

Let $g = 1$. Then T^*_{rel} is the trivial bundle, and $R + S \geq 1$.

for $k = 0$: L'_k is trivial. If $S = 0$ then $L'_k \otimes C$ is trivial, if $S \geq 1$ then $\deg\,(L'_k \otimes C)|_{S(\mathbf{w})\times\{\mathbf{u}\}} < 0$ so $H^0\left((L'_k \otimes C)|_{S(\mathbf{w})\times\{\mathbf{u}\}}\right) = \{0\}$.

for $k = 2$: If $S = 0$ then L'_k is trivial, if $S \geq 1$ then $\deg L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}} = S \geq 1$ and so $\dim H^0\left(L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}}\right) = \deg L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}} = S$. $L'_k \otimes C$ is trivial.

for $k \geq 4$: $\deg L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}}\,, \deg\,(L'_k \otimes C)|_{S(\mathbf{w})\times\{\mathbf{u}\}} \geq R + S \geq 1$.

Let $g \geq 2$.

for $k = 0$: L'_k is trivial. If $S = 0$ then $L'_k \otimes C$ is trivial, if $S \geq 1$ then $\deg\, (L'_k \otimes C)|_{S(\mathbf{w})\times\{\mathbf{u}\}} < 0$.

for $k = 2$: If $S = 0$ then $L'_k = T^*_{\mathrm{rel}}$ so $\dim H^0\left(L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}}\right) = g$, if $S \geq 1$ then $\deg L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}} = 2(g-1)+S \geq 2g-1$ and so $\dim H^0\left(L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}}\right) = \deg L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}} - g + 1 = g - 1 + S$. $L'_k \otimes C = T^*_{\mathrm{rel}}$.

for $k \geq 4$: $\deg L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}}\, , \deg\, (L'_k \otimes C)|_{S(\mathbf{w})\times\{\mathbf{u}\}} \geq 2g-1$. $\square$

Theorem 5.3 (main theorem) *We have isomorphisms*

$$\begin{array}{ccc} S_k(\Upsilon) & \simeq & S_k(\Gamma)\otimes\mathcal{P}^{\mathbb{C}} \\ \cap & \circlearrowleft & \cap \\ M_k(\Upsilon) & \simeq & M_k(\Gamma)\otimes\mathcal{P}^{\mathbb{C}} \\ {}_{\#}\searrow & \circlearrowleft & \swarrow_{\mathrm{id}\otimes\#} \\ & M_k(\Gamma) & \end{array} .$$

Proof: By lemma 5.2 after maybe replacing W and U_ρ , $\rho = 1, \dots, R$, V_σ , $\sigma = 1, \dots, S$, by smaller open neighbourhoods of $\mathbf{a} \in \mathcal{T}_g$ resp. $e_\rho, s_\sigma \in X$ there exist $F_1, \dots, F_r \in H^0\,(L'_k \otimes C) \hookrightarrow H^0\,(L'_k)$ and $F_{r+1}, \dots, F_{r'} \in H^0\,(L'_k)$ such that $\left(F_\rho|_{S(\mathbf{w})\times\{\mathbf{u}\}}\right)_{\rho\in\{1,\dots,r\}}$ is a basis of the $\mathbb{C}$-vectorspace $H^0\left((L'_k \otimes C)|_{S(\mathbf{w})\times\{\mathbf{u}\}}\right)$ and $\left(F_\rho|_{S(\mathbf{w})\times\{\mathbf{u}\}}\right)_{\rho\in\{1,\dots,r'\}}$ is a basis of the $\mathbb{C}$-vectorspace $H^0\left(L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}}\right)$ for all $(\mathbf{w}, \mathbf{u}) \in W \times \mathcal{U}$. Now define

$$f_\rho := F_\rho|_{S(\widetilde{\mathbf{a}})\times\{\widetilde{E}\}} \in H^0\left(L'_k|_{S(\widetilde{\mathbf{a}})\times\{\widetilde{E}\}}\right) ,$$

$\rho = 1, \dots, r'$. Then

$$f_1, \dots, f_r \in H^0\left((L'_k \otimes C)|_{S(\widetilde{\mathbf{a}})\times\{\widetilde{E}\}}\right) ,$$

$\left(f_1^\#, \dots, f_r^\#\right)$ is a basis of $H^0\left(\left((L'_k \otimes C)|_{S(\widetilde{\mathbf{a}})\times\{\widetilde{E}\}}\right)^\#\right)$, and $\left(f_1^\#, \dots, f_{r'}^\#\right)$ is a basis of $H^0\left(\left(L'_k|_{S(\widetilde{\mathbf{a}})\times\{\widetilde{E}\}}\right)^\#\right)$.

$$\left((L'_k \otimes C)|_{S(\widetilde{\mathbf{a}})\times\{\widetilde{E}\}}\right)^\# = (L'_k \otimes C)|_{S(\mathbf{a})\times\{E\}}$$

and

$$\left(L'_k|_{S(\widetilde{\mathbf{a}})\times\{\widetilde{E}\}}\right)^\# = L'_k|_{S(\mathbf{a})\times\{E\}} .$$

One obtains the result combining this with lemmas 1.5 and 5.1 . $\square$

Now we treat the case $2 \not| k$ *and* $-1 \notin \Gamma$ *.*

Let $s_1, \ldots, s_{S'} \in X$ be the even and $s_{S'+1}, \ldots, s_S \in X$ the odd cusps of $\Gamma \backslash H$.

We define the holomorphic line bundle $L_k \to \Xi_g|_W \times \mathcal{U}$ by the transition functions $\phi_{0\rho}(z) := (z - u_\rho)^{-k-1+2\left\lceil \frac{k+n_\rho}{2n_\rho} \right\rceil}$, $\rho = 1, \ldots, R$, $\psi_{0\sigma}(z) := (z - v_\sigma)^{-k}$, $\sigma = 1, \ldots, S'$ and $\psi_{0\sigma}(z) := (z - v_\sigma)^{-k+1}$, $\sigma = S' + 1, \ldots, S$.

Furthermore we define the holomorphic line bundle $C \to \Xi_g|_W \times \mathcal{U}$ by the transition functions $\phi_{0\rho}(z) := 1$, $\rho = 1, \ldots, R$, $\psi_{0\sigma}(z) := z - v_\sigma$, $\sigma = 1, \ldots, S'$, and $\psi_{0\sigma}(z) := 1$, $\sigma = S' + 1, \ldots, S$. Clearly $\deg C|_{S(\mathbf{w}) \times \{\mathbf{u}\}} = -S'$ for all $(\mathbf{w}, \mathbf{u}) \in W \times \mathcal{U}$, and we identify the sections of C with ordinary holomorphic functions on $\Xi_g|_W \times \mathcal{U}$ vanishing on $\{z = v_\sigma\}$, $\sigma = 1, \ldots, S'$.

Now let $F := \mathcal{F}^{\#}$, which is an ordinary holomorphic line bundle on $X = S(\mathbf{a})$. Then obviously $M_k(\Gamma) = H^0(F)$, and a straight forward calculation similar to the proof of lemma 5.1 shows that $F^{\otimes 2} \simeq \left({T^*_{\mathrm{rel}}}^{\otimes k} \otimes L_k \right)\Big|_{S(\mathbf{a}) \times \{E\}}$. So after maybe replacing W and U_ρ , $\rho = 1, \ldots, R$, V_σ , $\sigma = 1, \ldots, S$, by smaller open neighbourhoods of $\mathbf{a} \in \mathcal{T}_g$ resp. $e_\rho, s_\sigma \in X$ we may assume that there exists a unique line bundle $L'_k \to W \times \mathcal{U}$ such that $F \simeq L'_k|_{S(\mathbf{a}) \times \{E\}}$ and ${L'_k}^{\otimes 2} = {T^*_{\mathrm{rel}}}^{\otimes k} \otimes L_k$.

Lemma 5.4 *We have isomorphisms* $\mathcal{F} \simeq L'_k|_{S(\widetilde{\mathbf{a}}) \times \{\widetilde{E}\}}$ *and so* $M_k(\Upsilon) \simeq H^0\left(L'_k|_{S(\widetilde{\mathbf{a}}) \times \{\widetilde{E}\}} \right)$ *, the last isomorphism mapping* $S_k(\Upsilon)$ *to* $H^0\left((L'_k \otimes C)|_{S(\widetilde{\mathbf{a}}) \times \{\widetilde{E}\}} \right)$ *.*

Proof: By the same method as in the proof of lemma 5.1 one shows that

$$\mathcal{F}^{\otimes 2} \simeq \left({T^*_{\mathrm{rel}}}^{\otimes k} \otimes L_k \right)\Big|_{S(\widetilde{\mathbf{a}}) \times \{\widetilde{E}\}} = \left(L'_k|_{S(\widetilde{\mathbf{a}}) \times \{\widetilde{E}\}} \right)^{\otimes 2} .$$

Since $\mathcal{F}^{\#} = F \simeq L'_k|_{S(\mathbf{a}) \times \{E\}} = \left(L'_k|_{S(\widetilde{\mathbf{a}}) \times \{\widetilde{E}\}} \right)^{\#}$ the first assertion is a trivial consequence of the lemma 5.5 below.

For proving the last statement one just has to observe that if $f \in M_k(\Upsilon) = H^0(\mathcal{F})$ then $f \in S_k(\Upsilon)$ iff f vanishes in the local $\mathcal{P}$-chart Ψ_σ at 0 for all $\sigma = 1, \ldots, S'$, where Ψ denotes the local $\mathcal{P}$-chart of $\mathcal{X}$

at the cusp s_σ , $\sigma = 1, \dots, S'$. $\square$

Lemma 5.5 *Let $\mathcal{E}$ be a holomorphic $\mathcal{P}$- line bundle over the complex $\mathcal{P}$-manifold $\mathcal{M} = (M, \mathcal{S})$, $n \in \mathbb{N}$ and F be a holomorphic line bundle over M such that $F^{\otimes n} = \mathcal{E}^\#$. Then there exists an up to isomorphism unique $\mathcal{P}$-line bundle $\mathcal{F}$ over $\mathcal{M}$ such that $\mathcal{F}^\# = F$ and $\mathcal{F}^{\otimes n} \simeq \mathcal{E}$.*

Proof: Let $\mathcal{E}$ and F be given by the local trivializations on $U_i \subset M$ open, $i \in I$, $M = \bigcup_{i \in I} U_i$, with $\mathcal{P}$- transition functions $\varphi_{ij} \in \mathcal{S}(U_i \cap U_j)$ resp. transition functions $\psi_{ij} \in \mathcal{O}(U_i \cap U_j)$, $i, j \in I$. Without loss of generality we may even assume that $\mathcal{S}|_{U_i} \simeq \mathcal{O}_{U_i} \otimes \mathcal{P}$ for all $i \in I$.

Since $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$, $z \mapsto z^n$ is locally biholomorphic there exist unique $\widetilde{\psi_{ij}} \in \mathcal{S}(U_i \cap U_j) = \mathcal{O}(U_i \cap U_j) \otimes \mathcal{P}^{\mathbb{C}}$ such that $\widetilde{\psi_{ij}}^\# = \psi_{ij}$ and $\psi_{ij}^n = \varphi_{ij}$, $i, j \in I$. Now for proving existence define $\mathcal{F}$ via the local trivializations $\mathcal{S}|_{U_i}$ together with transition functions $\widetilde{\psi_{ij}} \in \mathcal{S}(U_i \cap U_j)$. For proving uniqueness let $\mathcal{F}'$ be another holomorphic $\mathcal{P}$- line bundle on $\mathcal{M}$ with body F and $\mathcal{F}'^{\otimes n} \simeq \mathcal{E}$. After maybe some refinement of the open cover $M = \bigcup_{i \in I} U_i$ we may assume without loss of generality that also $\mathcal{F}'$ admits local trivializations $\mathcal{S}|_{U_i}$ together with $\mathcal{P}$-transition functions $\varepsilon_{ij} \in \mathcal{S}(U_i \cap U_j)$ such that $\varepsilon_{ij}^\# = \psi_{ij}$ and $\varepsilon_{ij}^n = \varphi_{ij}$, $i, j \in I$. Therefore $\varepsilon_{ij} = \widetilde{\psi_{ij}}$, $i, j \in I$, and so $\mathcal{F}' \simeq \mathcal{F}$. $\square$

Again

$$\begin{aligned} \deg L'_k|_{S(\mathbf{w}) \times \{\mathbf{u}\}} &= \frac{k}{2} \deg T^*_{\mathrm{rel}}|_{S(\mathbf{w}) \times \{\mathbf{u}\}} + \frac{1}{2} \deg L_k|_{S(\mathbf{w}) \times \{\mathbf{u}\}} \\ &= k(g-1) + \frac{k+1}{2} R - \sum_{\rho=1}^{R} \left\lceil \frac{k + n_\rho}{2 n_\rho} \right\rceil \\ &\quad + \frac{k-1}{2} S + \frac{S'}{2} \end{aligned} \tag{2}$$

is independent of the point $(\mathbf{w}, \mathbf{u}) \in W \times \mathcal{U}$, so automatically $2 \,|\, S'$.

Lemma 5.6 *Let $g \leq 1$ or $k \geq 3$. Then*

$$\dot{\bigcup}_{(\mathbf{w}, \mathbf{u}) \in W \times \mathcal{U}} H^0 \left(L'_k|_{S(\mathbf{w}) \times \{\mathbf{u}\}} \right)$$

is a holomorphic vector bundle over $W \times \mathcal{U}$ containing

$$\dot{\bigcup}_{(\mathbf{w}, \mathbf{u}) \in W \times \mathcal{U}} H^0 \left((L'_k \otimes C)|_{S(\mathbf{w}) \times \{\mathbf{u}\}} \right)$$

as a holomorphic sub vector bundle.

Proof: same as the proof of lemma 5.2 in the case $2|k$, now using formula (2) .

Let $g = 1$.

for $k = 1$: If $S' = 0$ then $L'_k = L'_k \otimes C$ and $L'^{\otimes 2}_k$ is trivial. So either L'_k is trivial or $H^0\left(L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}}\right) = \{0\}$. If $S' \geq 2$ then
$\deg L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}} = \frac{S'}{2} \geq 1$ and
$\deg\,(L'_k \otimes C)|_{S(\mathbf{w})\times\{\mathbf{u}\}} = -\frac{S'}{2} < 0$.

for $k \geq 3$: If $k = 3$, $S = 0$ and all $n_\rho = 2$ then $L'_k = L'_k \otimes C$ and $L'^{\otimes 2}_k$ is trivial. In all other cases $\deg L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}}\,, \deg\,(L'_k \otimes C)|_{S(\mathbf{w})\times\{\mathbf{u}\}} \geq 1$.

Let $g \geq 2$.

for $k \geq 3$: $\deg L'_k|_{S(\mathbf{w})\times\{\mathbf{u}\}}\,, \deg\,(L'_k \otimes C)|_{S(\mathbf{w})\times\{\mathbf{u}\}} \geq 3(g-1) \geq 2g-1$. $\square$

Theorem 5.7 (main theorem) *If $g \leq 1$ or $k \geq 3$ then we have isomorphisms*

$$\begin{array}{ccc} S_k(\Upsilon) & \simeq & S_k(\Gamma) \otimes \mathcal{P}^{\mathbb{C}} \\ \cap & \circlearrowleft & \cap \\ M_k(\Upsilon) & \simeq & M_k(\Gamma) \otimes \mathcal{P}^{\mathbb{C}} \\ \# \searrow & \circlearrowleft & \swarrow_{\mathrm{id}\otimes\#} \\ & M_k(\Gamma) & \end{array} .$$

Proof: similar to the case $2|k$ now using lemmas 1.5 , 5.4 and 5.6 . $\square$

In the case $g \geq 2$ and $k = 1$ the theorem indeed fails to be true. Here a counter example:

Let $U \subset \mathbb{C}$ be an open connected neighbourhood of 0 , $\Xi \to U$ be a holomorphic family of compact RIEMANN surfaces $S(w)$ of genus $g \geq 1$ and $L \to \Xi$ be a holomorphic line bundle (so automatically $\deg L|_{S(w)}$ is independent of the point w) having $\dim H^0\left(L|_{S(w)}\right) < \dim H^0\left(L|_{S(0)}\right)$ for all $w \in U \setminus \{0\}$ (so automatically $0 \leq \deg L|_{S(w)} \leq 2(g-1)$). Then there exists $f \in \dim H^0\left(L|_{S(0)}\right)$ with the following property:

> f admits **no** extension to 'compact Riemann surfaces nearby' , which means there exists no pair (U', F) , where $U' \subset U$ is an open neighbourhood of 0 and $F \in H^0\left(L|_{U'}\right)$ such that $F|_{S(w)} = f$.

Now let $z_0 \in S(0)$ be arbitrary. After maybe replacing U by a smaller open neighbourhood of 0 we may fix a common local chart of all $S(w)$, $w \in U$, being a local coordinate neighbourhood of $S(0)$ at z_0 . Via this common local chart we may regard z_0 as a point of $S(w)$ for each $w \in U$. Let $R \to \Xi$ be the holomorphic line bundle such that for each $w \in U$ the holomorphic sections of $R|_{S(w)}$ are the meromorphic functions on $S(w)$ which are holomorphic on $S(w) \setminus \{z_0\}$ and have a pole at $z_0 \in S(w)$ of order at most $d := 2g - 1 - \deg L|_{S(w)}$. So the holomorphic functions on $S(w)$ can be regarded as holomorphic sections of R vanishing at z_0 of order at least d .

Clearly $\deg(L \otimes R)|_{S(w)} = 2g - 1$, and so $\dim H^0\left((L \otimes R)|_{S(w)}\right) = g$ is independent of the point $w \in U$. So again by theorem 5 in section 10.5 of [4] we see that

$$\dot{\bigcup} H^0\left((L \otimes R)|_{S(w)}\right)$$

is a vector bundle of rank g over U . After maybe replacing U by a smaller open neighbourhood of 0 we can assume that there exists a frame $(F_1, \ldots, F_g) \in H^0(L \otimes R)^{\oplus g}$. So there exists $F \in H^0(L \otimes R)$ having $F|_{S(0)} = f$, of course

$$\Phi(F)(w) := \begin{pmatrix} F(w, z_0) \\ \vdots \\ \partial_z^{d-1} F(w, z_0) \end{pmatrix} \in \mathcal{O}(U)^{\oplus d}$$

must have an isolated zero at $w = 0$.

Lemma 5.8 *There exists $N \in \mathbb{N}$ such that $\mathrm{ord}_0\, \Phi(F) \leq N$ for all extensions $F \in H^0(L \otimes R)$ of f .*

Proof: Let $F \in H^0(L \otimes R)$ having $F|_{S(0)} = f$ be given. Then all other extensions of f in $H^0(L \otimes R)$ are given by $\widetilde{F} = F + w \sum_{j=1}^{g} \varphi_j F_j$ with $\varphi_j \in \mathcal{O}(U)$, $j = 1, \ldots, g$.

$$\begin{aligned} \Phi\left(\widetilde{F}\right)(w) &= \begin{pmatrix} F(w, z_0) \\ \vdots \\ \partial_z^{d-1} F(w, z_0) \end{pmatrix} \\ &\quad + w \begin{pmatrix} F_1(w, z_0) & \ldots & F_g(w, z_0) \\ \vdots & & \\ \partial_z^{d-1} F_1(w, z_0) & \ldots & \partial_z^{d-1} F_g(w, z_0) \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_g \end{pmatrix} . \end{aligned}$$

After trigonalization of the matrix using total pivot search with respect to the order at $w=0$ one obtains up to multiplication with some element of $GL(d,\mathcal{O}(U))$ and permutation of the φ_j's

$$\Phi\left(\widetilde{F}\right)(w)=\begin{pmatrix} H_1\\ \vdots \\ H_d\end{pmatrix}+w\left(\begin{array}{ccc} a_1 & & \\ & \ddots & b_{ij} \\ 0 & & a_{d'} \\ \hline & 0 & \end{array}\right)\begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_g\end{pmatrix},$$

$d'\in\{0,\dots,\min(d,g)\}$, $H_i,a_i,b_{ij}\in\mathcal{O}(U)$, $\operatorname{ord}_0 b_{ij}\geq\operatorname{ord}_0 a_i$, $i=1,\dots,d'$, $j=i+1,\dots,g$.

Since the linear system $\Phi\left(\widetilde{F}\right)=0$ has no solution $(\varphi_1,\dots,\varphi_g)\in\mathcal{O}_{U,0}^{\oplus g}$ there exists at least one $i\in\{1,\dots,g\}$ such that $H_i\neq 0$ if $i\geq d'+1$ resp. $\operatorname{ord}_0 H_i\leq\operatorname{ord}_0 a_i$ if $i\leq d'$. So take $N:=\operatorname{ord}_0 H_i$. $\square$

Now let $\mathcal{P}:=\mathbb{R}[X]\big/\left(X^{N+1}=0\right)$ and $\widetilde{w}:=\overline{X}\in\mathcal{P}$. Then $\widetilde{w}\in_{\mathcal{P}}U$ with $\widetilde{w}^{\#}=0$.

Lemma 5.9 *There exists* **no** $\widetilde{f}\in H^0\left(L|_{S(\widetilde{w})}\right)$ *such that* $\widetilde{f}^{\#}=f$.

Proof: Assume $\widetilde{f}\in H^0\left(L|_{S(\widetilde{w})}\right)$ having $\widetilde{f}^{\#}=f$. Then $\widetilde{f}$ can be regarded as an element of $H^0\left((L\otimes R)|_{S(\widetilde{w})}\right)$ having a zero of order at least d in $z_0\in_{\mathcal{P}}S(\widetilde{w})$. So there exist $\overline{a_1},\dots,\overline{a_g}\in\mathcal{P}^{\mathbb{C}}$, $a_1,\dots,a_g\in\mathbb{C}[X]$, such that

$$\widetilde{f}=\sum_{j=1}^{g}\overline{a_j}\,F_j|_{S(\widetilde{w})}\ .$$

Since $f=\widetilde{f}^{\#}=\sum_{j=1}^{g}\overline{a_j}^{\#}\,F_j|_{S(0)}$, we see that $F:=\sum_{j=1}^{g}a_j(w)F_j\in H^0(L\otimes R)$ is an extension of f . Finally since $F|_{\widetilde{w}}=\widetilde{f}\in H^0\left((L\otimes R)|_{S(\widetilde{w})}\right)$ vanishes at $z_0\in_{\mathcal{P}}S(\widetilde{w})$ of order at least d we see that $\Phi(F)(\widetilde{w})=0$, and so $\Phi(F)$ has a zero at $w=0$ of order at least $N+1$, which is a contradiction to lemma 5.8 . $\square$

Apply this to a suitable family of theta characteristics $L\to\Xi$, $g\geq 2$, which means a holomorphic line bundle L having $L^{\otimes 2}=T^*_{\mathrm{rel}}$, see [1] , appendix B . $\dim H^0\left(L|_{S(w)}\right)$ is only constant mod 2 . Since any holomorphic family of compact RIEMANN surfaces of genus g can be written as the pullback of a holomorphic map into $\mathcal{T}_g$, and $\mathcal{T}_g$, $g\geq 2$, can be written as the moduli space of certain cocompact lattices in G , see [7] and example 2.3 (ii) , we know that there exists a smooth map $\varphi:U\to G^{2g}$, $w\mapsto(A_1(w),B_1(w),\dots,A_g(w),B_g(w))$ such that

(i) all $A_1(w), B_1(w), \ldots, A_g(w), B_g(w) \in G$ are hyperbolic generating a cocompact lattice Γ_w without elliptic elements and $-1 \notin \Gamma_w$,

(ii) $S(w) = \Gamma_w \backslash H$,

(iii) $L|_{S(w)}$ is obtained by the identification $(\gamma z, S) \sim (z, j(\gamma, z)S)$, $\gamma \in \Gamma_w$, in $H \times \mathbb{C}$.

In particular (iii) is guaranteed by lemma 11.1 in [7] . So $\Upsilon := \Gamma_{\widetilde{w}}$ is a $\mathcal{P}$-lattice of G with body $\Gamma := \Gamma_0$, and by lemma 5.4 we have
$M_1(\Upsilon) = S_1(\Upsilon) \simeq H^0\left(L|_{S(\widetilde{w})}\right)$.

6 Body $SL(2, \mathbb{Z})$

Let Υ be a $\mathcal{P}$-lattice of G with body $SL(2, \mathbb{Z})$. Then of course this special case is easier to handle than the general theory. First of all $M_k(\Upsilon) = 0$ if $2 \not| k$. So we can restrict our investigations the case $2|k$.

Let $\mathcal{X} = (X, \mathcal{S})$, $X := SL(2, \mathbb{Z}) \backslash H \cup \{\overline{\infty}\}$, be the $\mathcal{P}$- Riemann surface given by the construction of section 3 . Since X is of genus $g = 0$ we may identify $X \simeq \mathbb{P}^1$, and from corollary 4.4 we know that there exists a $\mathcal{P}$-isomorphism $\Phi : X \to_{\mathcal{P}} \mathcal{X}$ with $\Phi^{\#} = \mathrm{id}$. Let $\widetilde{e_1}, \widetilde{e_2} \in_{\mathcal{P}} \mathcal{X}$ be the two elliptic points of $\Upsilon \backslash H$ with bodies $e_1 := \overline{i}$ resp. $e_2 := \overline{e^{\frac{2}{3}\pi i}} \in X$, which are precisely the elliptic points of $SL(2, \mathbb{Z}) \backslash H$, and let $\widetilde{s} \in_{\mathcal{P}} \mathcal{X}$ be the cusp of $\Upsilon \backslash H$ with body $s := \overline{\infty} \in X$, which is the cusp of $SL(2, \mathbb{Z}) \backslash H$. One knows that $\mathrm{Aut}(X) \simeq SL(2, \mathbb{C}) / \{\pm 1\}$.

Lemma 6.1 *There exists a unique $g \in_{\mathcal{P}} SL(2, \mathbb{C})$ such that $g^{\#} = 1$, $ge_1 = \Phi^{-1}(\widetilde{e_1})$, $ge_2 = \Phi^{-1}(\widetilde{e_2})$ and $gs = \Phi^{-1}(\widetilde{s})$.*

Proof: A straight forward computation shows that

$$G \to X^3 \, , \, g \mapsto (ge_1, ge_2, gs)$$

is locally biholomorphic at 1 , which proves the lemma. $\square$

Now a simple calculation using the local $\mathcal{P}$-charts of $\mathcal{X}$ given in section 3 shows that $\Phi \circ g$ uniquely lifts to a $\mathcal{P}$-automorphism $\Omega : H \to H$ having $\Omega^{\#} = \mathrm{id}$ such that

$$\begin{array}{ccc} H & \xrightarrow{\Omega} & H \\ \pi \downarrow & \circlearrowleft & \downarrow \Pi \\ X & \xrightarrow[\Phi \circ g]{} & \mathcal{X} \end{array} ,$$

so automatically for all $\gamma \in \Upsilon$

$$\begin{array}{ccc} H & \xrightarrow{\Omega} & H \\ \gamma^{\#} \downarrow & \circlearrowleft & \downarrow \gamma \\ H & \xrightarrow[\Omega]{} & H \end{array} .$$

Theorem 6.2 *For all $k \in 2\mathbb{N}$*

$$\Psi_k : M_k(\Upsilon) \to M_k(SL(2,\mathbb{Z})) \otimes \mathcal{P}^{\mathbb{C}} , \; f \mapsto f|_\Omega$$

is a $\mathcal{P}^{\mathbb{C}}$-module isomorphism mapping $S_k(\Upsilon)$ to $S_k(SL(2,\mathbb{Z})) \otimes \mathcal{P}^{\mathbb{C}}$, and all Ψ_k , $k \in 2\mathbb{N}$, glue together to an isomorphism of $2\,\mathbb{N}$-graded $\mathcal{P}^{\mathbb{C}}$-algebras

$$\begin{array}{ccccc} \bigoplus_{k\in 2\mathbb{N}} M_k(\Upsilon) & & \to & & \bigoplus_{k\in 2\mathbb{N}} M_k(SL(2,\mathbb{Z})) \otimes \mathcal{P}^{\mathbb{C}} \\ & {}_{\#}\searrow & \circlearrowleft & \swarrow_{\mathrm{id}\otimes\#} & \\ & & \bigoplus_{k\in 2\mathbb{N}} M_k(SL(2,\mathbb{Z})) & & \end{array} .$$

Proof: We use the notation of section 3 and the even case of section 5 . Therefore we have to identify $X = X \times \left\{\widetilde{E}\right\}$ with $\mathcal{X}$ via Φ . $\Phi \circ g$ induces an identification $(T^*\mathcal{X})^{\otimes \frac{k}{2}} \simeq (T^*X)^{\otimes \frac{k}{2}}$, which restricted to $SL(2,\mathbb{Z})\backslash H \subset X$ is given by $f \mapsto f|_\Omega$, and g as $\mathcal{P}$-automorphism of X induces identifications $L_k|_{X\times\{\widetilde{E}\}} \xrightarrow{\sim} L_k|_{X\times\{E\}}$ and $C|_{X\times\{\widetilde{E}\}} \xrightarrow{\sim} C|_{X\times\{E\}}$ as holomorphic $\mathcal{P}$-line bundles given by $f \mapsto f(gz)$, where we use a local coordinate z on $X \simeq \mathbb{P}^1$. Since $\mathcal{F} \simeq \left.\left(T^{*\,\otimes\frac{k}{2}}_{\mathrm{rel}} \otimes L_k\right)\right|_{X\times\{\widetilde{E}\}}$ by lemma 5.1 with body $\mathcal{F} \simeq \left.\left(T^{*\,\otimes\frac{k}{2}}_{\mathrm{rel}} \otimes L_k\right)\right|_{X\times\{E\}}$ and $T^*_{\mathrm{rel}}|_{X\times\{\widetilde{E}\}} = T^*_{\mathrm{rel}}|_{X\times\{E\}} = T^*X$ the claim follows. $\square$

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